Inquiry in University Mathematics Teaching and Learning

The PLATINUM Project

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Editors

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CHAPTER 18

Experience in Implementing IBME at the Borys Grinchenko Kyiv University

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18.1. Development of an IBME Community at BGKU

In recent years, Borys Grinchenko Kyiv University has faced the problem of low motivation of future students in choosing mathematics programmes, and in teaching—with the problem of involving students in active learning through the use of new methods of teaching mathematics. One of the ways to tackle these problems is the use of innovative pedagogy and educational technology by mathematics teachers, which stimulate students' motivation to study mathematics and their involvement in the learning process. This includes inquiry-based mathematics education (IBME). With this in mind, an educational community of mathematics teachers was created at the university to acquaint teachers with IBME and the peculiarities of its use. The purpose of creating such a community was: to acquaint teachers with the concept, types, and cyclic structure of inquiry; examples of use; discussion of the use of inquiry in the teaching and learning of mathematics at the university. Initially, theoretical problems were analysed, the experience of creating educational communities was studied as well as forms of their activities to attract university teachers to participate in the community.

Quarantine measures in response to the COVID-19 pandemic led to significant changes in the lives of everyone in all countries of the world during 2020, including the organisation of the educational process in both secondary education and universities. The special conditions imposed on the work of educators have made apparent that one of the factors in the progress of educational reforms depends on the individual and collective ability of teachers to contribute to the transformation of the educational process. One of the ways to improve the quality of the educational process is the constant exchange of experience between teachers, discussion of existing pedagogical problems, analysis of best innovative educational practices, their implementation, and further discussion in the community of educators who are experts in a particular field. Therefore, the introduction and dissemination of such professional communities are relevant and important. Educational communities create favourable conditions and motivation for constant professional development of academic staff in higher education institutions.

Research confirms that the activities of educational communities have a positive impact on the results of the educational process for both teachers and students (Hattie, 2012; Hord, 1997; Jaworski, 2005; Marzano, 2003; Solomatin, 2015; Brodie & Chimhande, 2020). When teachers are part of the professional educational community, it reduces their isolation (certainly during quarantine periods), increases their commitment to the mission and understanding of the goals of the institution, creates conditions to support joint responsibility for the formation and development of professional competencies of students, and supports positive motivation to improve skills. This allows us to share the best teaching practices and expands the understanding of the content of educational material and the new role of the teacher in the digital transformation of education.

The term "educational community" has been introduced in Ukrainian pedagogy only recently. It is often associated with cyberspace (Maluhin & Aristova, 2020; Levchenko, 2020), while foreign studies do not narrow this concept to the web interface (Tönnies, 2002) and speak of 'community of practice' (Wenger, 1998) and 'community of inquiry' (Jaworski, 2005). Educational communities allow all participants to develop both personally and professionally. Training depends on the educational sector and tasks of the community. Participation in a community means, first and foremost, access to its resources, which can be both tangible and intangible (Tönnies, 2002). Some educators view the educational community as a dissemination of classroom practice in a community, using the resources of that community, both material and human. Others understand the educational community as involving specialists in educational institutions to improve the curriculum and learning objectives for students' educational activities (Hersi et al., 2016). For some educators, community activities involve the mutual learning of students, academic staff, and administrators, through the use of various organisational forms, pedagogical, and digital technologies.

Identification of these four important components was the first step in the formation of the community (Figure 18.1). These components were declared "important" a priori and served as a (normative) guideline/aim for the process of forming a community. The development of the community at BGKU confirmed the importance of each of these components.

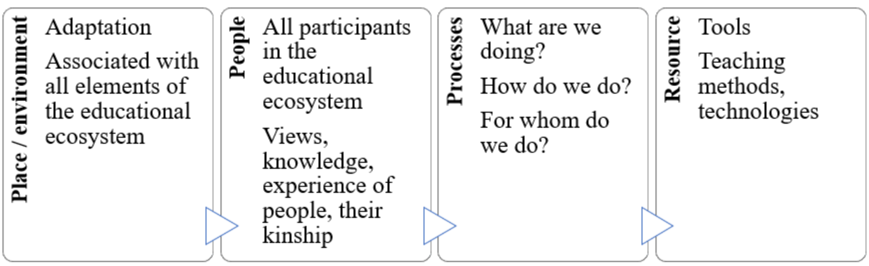


FIGURE 18.1. Components of communities.

Initially, the community consisted only of BGKU faculty who are members of the PLATINUM project. One of the main conditions for the creation and functioning of the educational community is the common goal of its members. Because the established community is engaged in the study of IBME, the name of the community was narrowed to IBME-community, a group of people that explore and disseminate inquiry-based mathematics education. The main target of the IBME-community in BGKU was implementing inquiry-based learning in high school mathematics education. Over time, the community was joined by mathematics teachers from other universities, employees of the Institute of Mathematics (National Academy of Sciences).

The main stages of forming such a community of teachers in BGKU were:

- holding an organisational meeting with teachers;
- developing a community promotion strategy;

- creating a questionnaire for the IBME survey;
- conducting a survey;
- conducting seminars on IBME;
- defining the main features of the professional community;
- creating a site for the community of mathematicians and a page on the Wiki portal;
- creating a Facebook page for the community;
- facebook page support, site creation, and support;
- activity planning and community development.

The initial survey of community members involved determining the respondents' experience in teaching, the list of subjects they teach, the educational institution where they work, their educational needs, and problems in teaching mathematics in university.

Community activities include discussion of open lectures, brainstorming in solving didactic problems, analysis of scenarios for involving students in active learning in mathematics, pedagogical technologies for teaching students inquiry, reflection on the introduction of inquiry technologies in various types of classes—lectures, practical training, discussions, surveys of teachers and students, group work, workshops, discussion of ways to use digital technologies in teaching mathematics.

At the stage of integrating the academic staff into the community, practical seminars, and workshops were held, during which the following issues were discussed:

- The concept of "educational community." Features of community activities and their functions.
- STEM-education and innovative methods—problem-based learning, project-based learning, inquiry-based learning. Commonalities and differences between these approaches.
- Inquiry questions. Criteria for inquiry questions.
- The three-layer model of inquiry adopted in PLATINUM (Chapter 2). Inquiry in mathematics in the classroom using the 5E-model of instruction to develop students' research skills (Bybee et al., 2006)
- Examples of mathematical research environments in the Go-Lab online laboratory,[1] which is a tool for learning and using Inquiry-Based Science Education (IBSE) in practice.

During the study of practical aspects of creating and organising the work of the IBME-community of teachers at BGKU, an empirical method was used (initial and repeated questionnaires of teachers of higher education institutions), as well as analysis of the results. The questionnaire in electronic form was created and sent to all members of the BGKU community. The initial survey was conducted at the stage of community formation, and the second after six months of cooperation of teachers in the community. A total of 72 respondents took part in the survey. The purpose of the survey was: to determine the impact of teacher activities in the community to change its methods of teaching and learning mathematics associated with the use of inquiry in teaching mathematics. For example, the objective of one question was to find out "What teaching methods do you use most often in your pedagogical activity?" before participating in the community and during joint activities in the community. Therefore, whether community participation influenced the use of innovative pedagogical methods of teaching mathematics. The results of the survey showed the percentage of teachers who began to use project-based learning in teaching mathematics (at

[1] `www.golabz.eu`

the beginning of the community 30.6%, at the stage of community activity 50%), research (33.3% and 66.7% respectively), and inquiry-based learning approach (41.7% and 83.3%, respectively) (Figure 18.2).

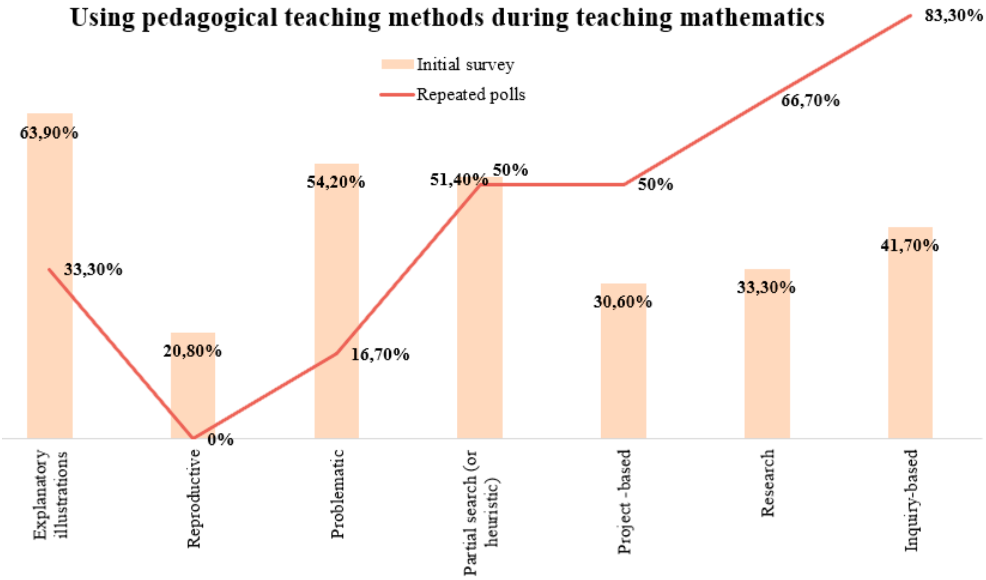


FIGURE 18.2. Teaching approaches reported within the community at the start of the project (the bar chart) and half a year later (the line graph).

Interesting are the changes in the forms of work that began to be used by members of the community with students in teaching mathematics. At the beginning of joint activities, teachers most often used the group form of work (70.8%) and individual (70.8%). After the exchange of experience and participation in workshops, the group form of student work became a priority (83.3%). At the same time, the percentage of using the individual form of work decreased by 20.8%.

One of the didactic techniques discussed during the community meetings was flipped classroom. The result of the exchange of experience and identification of the peculiarities of the organisation of this innovative pedagogical technology in the study of mathematics was an increase in the percentage of teachers (at the beginning of the community 50%, at the stage of forming community 66.7%), who began to use the flipped classroom in their professional activities, using digital tools. Members of the BGKU IBME-community defined their role in the use of IBME. The primary and secondary surveys showed a difference in priorities. Participants ranked the teacher's role from 1 (not important) to 7 (extremely important). As the result shows, teachers are interested in the stage of engagement, motivating students to inquiry activities, while planning itself has become less important, which demonstrates the willingness of teachers and students to use the method of "open inquiry" in this approach (Figure 18.3).

Analysing the role of the community in implementing IBME for teaching mathematics, all faculty members answered the question "Did you learn anything new about the inquiry approach after sharing experiences in the community?" with "Yes, a lot of interesting cases and tricks." To the question "Did the experience of working in the community allow you to share your work, research, observations?" two-thirds answered "Yes," and one-third "Partially."

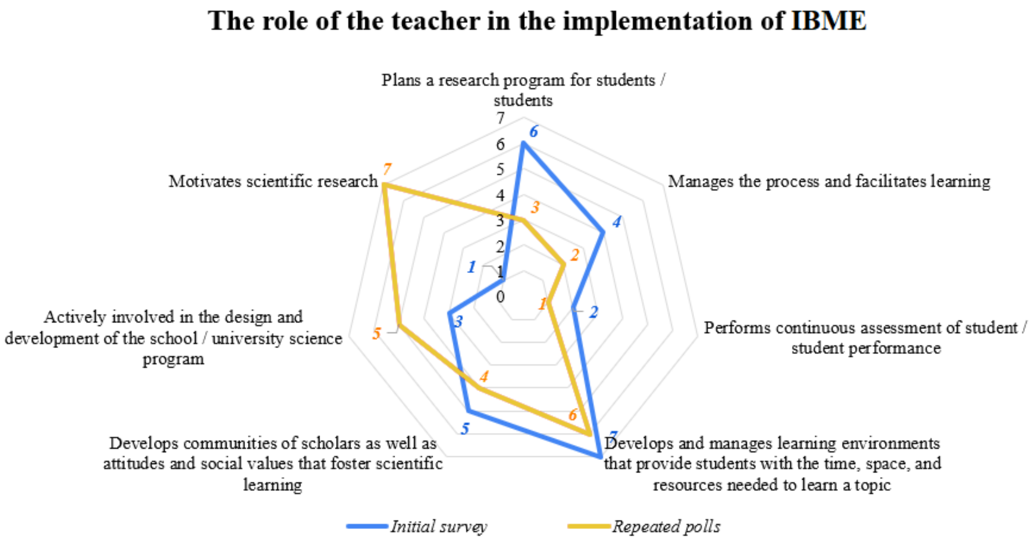


FIGURE 18.3. The role of the teacher in the implementation of IBME.

One of the stages of activity in the IBME-community consists of the creation and discussion of case studies, exchange of experience in implementing IBME, conducting open classes, development of templates for research tasks, creation of a database of modelling tasks, discussion of students' academic achievements and their reflection on changes in educational activities during the implementation of IBME. Currently, community teachers are actively working at this stage. This allows exploring the impact of community functioning on the professional activities of teachers and the process of implementing IBME in teaching mathematics, on positive changes in motivation and interest of students in mathematics, the results of their academic achievements in mathematics. In the next section we describe one of the stages of activity in the IBME community, namely, a case on the use of IBME in the study of mathematical analysis (lecture description, description of the organisation of students' homework) to improve the conceptual understanding of mathematics and the formation of conceptual knowledge.

18.2. IBME for the Formation of Conceptual Knowledge During Teaching of Mathematical Analysis

18.2.1. Conceptual and Procedural Knowledge. A deep understanding of mathematics and the ability to use it in further professional activities require two types of knowledge: conceptual and procedural. First of all, let's find out what conceptual knowledge is. There are different interpretations of the term "conceptual knowledge." Most of them, despite some differences, agree that conceptual knowledge involves not only knowledge of individual concepts, facts, methods, but also an understanding of the relationships between them, seeing how some facts follow from others, the ability to see the key idea of one or another method, to feel in what contexts it can be useful, to apply it in problem-solving, etc. (Hiebert & Lefevre, 1986; Cobb, 1988; Byrnes & Wasik, 1991; Haapasalo & Kadijevich, 2000; Star, 2005). All these characteristics of conceptual mathematical knowledge are quite accurately conveyed by Hiebert and LeFevre's definition: "Conceptual knowledge is characterised most clearly as the knowledge that is rich in relationships. It can be thought of as a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information." In addition, this definition leads to an important conclusion for teaching mathematics, which Star (2005) emphasises: "The term conceptual knowledge has come to encompass not only what is known (knowledge of concepts) but also one way that concepts can be known (e.g., deeply and with rich connections)" (p. 408). That is, the learning process aimed at achieving conceptual understanding and the formation of conceptual knowledge requires a significant reorganisation of existing knowledge, not just its accumulation. From the point of view of Hiebert and Lefevre (1986), procedural knowledge—rules or algorithms—are represented mainly in symbolic form. Haapasalo and Kadijevich (2000, p. 141) see procedural knowledge as "dynamic and successful utilization of particular rules, algorithms or procedures within relevant representation form(s)."

The question of procedural knowledge vs. conceptual knowledge has been the focus of many researchers (e.g., Lauritzen, 2012). There are four main models of causal relationships between conceptual (C) and procedural (P) knowledge:

(1) *Genetic*: the presence of C automatically ensures obtaining P, but the formation of P does not ensure the formation of C.
(2) *Dynamic interaction*: the presence of P and problem-solving forms C, but the formation of C does ensure getting P.
(3) *Simultaneous activation*: the presence of P and problem-solving forms C, and the formed C, in turn, helps to obtain P.
(4) *Inactivation*: P and C are not related

Our teaching practice over the years shows that the third model improves our personal experiences. It follows that conceptual knowledge can both precede or influence procedural knowledge, and procedural knowledge can precede conceptual knowledge. However, conceptual knowledge without procedural knowledge is ineffective, and procedural knowledge without conceptual knowledge is superficial, and can lead to serious errors in the use of mathematics later, and can hardly be applied in unfamiliar contexts. Conceptual knowledge improves the student's ability to detect misuse of a method (procedure) or inconsistency of the method in a given situation, and to analyse and evaluate an answer (Lauritzen, 2012). Moreover, as shown in (Chappell & Killpatrick, 2003), students for whom the concept-based learning environment was created showed already during training much better results than students in the procedure-based learning environment. Both conceptual understanding of mathematics and procedural skills were assessed in this reference.

Thus, conceptual knowledge is a necessary component of teaching Mathematics and considerable attention must be given to its formation. But, unlike procedural knowledge, "conceptual knowledge is the most difficult to acquire. It's difficult because knowledge is never acquired *de novo*; a teacher cannot pour concepts directly into students' heads." (Willingham, 2009, p. 18). When teaching higher mathematics in the context of concept-based learning, we must provide conceptual understanding so that students

- understand which mathematical ideas are key and why;
- are aware of the systemic nature of mathematics and see the relationship of its areas;
- understand what ideas can be applied in a particular context, and understand the basic methods of mathematical proofs and the scope of their application; and
- can adapt prior experience to new problems, especially to nonstandard ones.

Let us note that it is much more difficult to assess the level of formation of conceptual knowledge and identify their gaps rather than procedural ones. To do this, it is necessary to select appropriate problems that require a systematic approach, application of previous experience, and analytical thinking of the student. During the teaching of mathematics it is necessary to evaluate the following learning outcomes:

- knowledge of mathematical concepts, statements, theorems, properties, features, methods, and ideas; the ability to apply the acquired knowledge and skills to solve educational and practical problems, when the method of such a solution must be found by herself/himself (*conceptual knowledge*);
- knowledge of the methods of activity that can be presented in the form of a system of actions (rules, algorithms); ability to perform already known actions following the learned rules, algorithms (*procedural knowledge*).

In Figure 18.4 are shown examples of problems that we proposed when studying Rolle's theorem (in the context of Mathematical Analysis).

Problem A (*procedural-oriented*):
Can Rolle's Theorem be applied to the function $f(x) = 1 - \sqrt[3]{x^2}$ on the interval $[-\frac{1}{2}, \frac{1}{2}]$?

Problem B (*conceptual-oriented*):
Can the equation $e^x = ax^2 + bx + c$, where $a, b, c \in \mathbb{R}$, have four roots?

FIGURE 18.4. Two problems related to Rolle's Theorem.

Although the formulation of both problems looks problematic ("can"), problem A is purely procedural. To obtain the answer it is sufficient to check the fulfilment of the conditions of Rolle's Theorem and to be able to find the derivative of the function by known rules. To solve problem B, you need (1) to find out whether the equation can have roots at all; (2) to form a hypothesis ("cannot"), and for this to resort to graphical interpretation, conditional experiment; and (3) to prove it (feel that the proof should be carried out by the method proof by contradiction, and use Rolle's Theorem). However, Task A could be made conceptual-oriented with the help of inquiry questions that encourage students to continue their research. For example: "If your answer to question A is negative, does this mean that there is no point at this interval where the derivative of the function is zero?" (Answer: no); "Is there an interval at which Rolle's theorem can be applied to this function?" (Answer: no).

The key to students' high academic achievements is *active learning*, which means the use of such methods and techniques that require students' conscious educational activities, involve them in the process of constructing new knowledge, research skills, and their use. Active student participation is a key factor that influences the success of the entire educational process in higher education. This is found in many studies and not exclusively in mathematics education. Active learning has been found to improve conceptual understanding (Laws et al., 1999), improving performance better than increasing study time (Redish et al., 1997). For example, an analysis of 225 studies comparing students' performance in active and traditional (passive-informative) learning in Science, Technology, Engineering, and Mathematics confirmed the effectiveness of active learning in STEM education. (Freeman et al., 2014).

Information can be obtained passively, but not understanding, because it requires the connection between prior and new knowledge. And this is possible only as a result of active mental actions. Learning based on memorising and using algorithms saves time but does not contribute to the formation of conceptual knowledge and the development of critical thinking. Below are three of the students' most typical opinions about this, expressed after six months of studying mathematical analysis.

> Now I see that nothing needs to be crammed in mathematics. Finally (!) I understood where many of the formulas we studied at school came from. At school, they just aimed at memorising and that's all.
>
> At the beginning of my studies, I tried to memorise definitions, theorems, formulas to reproduce when asked. But they did not keep in mind, because they were very unusual and incomprehensible. Fortunately, I soon concluded that the main thing is to understand concepts, facts, imagine them, look for and find convincing arguments. Living and studying have become much easier and more interesting.
>
> I really like the way we study. We reflect on new concepts and facts. But at the same time we can always count on the friendly help of the teacher. And it's very inspiring.

Thus, active learning strategies can act as a mechanism for the development of conceptual understanding of mathematical structures, creative thinking, research competencies, and meta-skills. One of such strategies is IBME.

18.2.2. Characteristics of the Discipline and the Cohort of Students. An example is a case of IBME implementation to form conceptual knowledge for the teaching of mathematical analysis to first-year students majoring in Mathematics, by a lecturer—Associate Professor of the Department of Computer Science and Mathematics, Ph.D. (Physical and Mathematical Sciences) Mariia Astafieva.

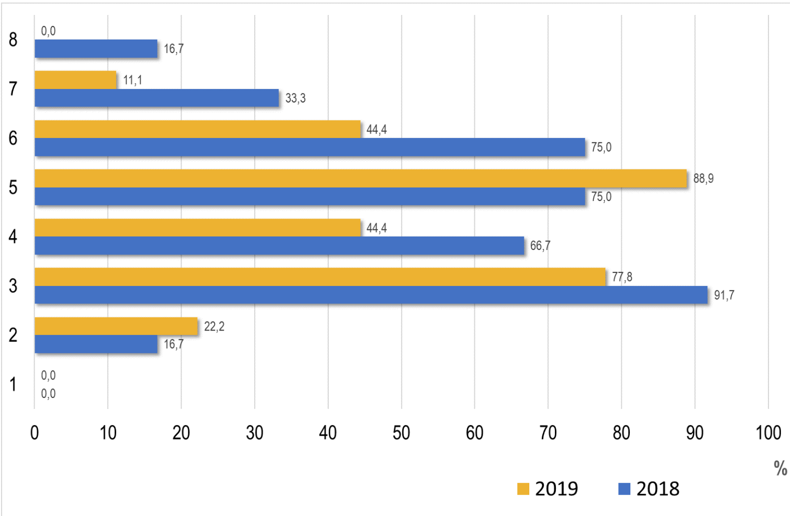
Mathematical Analysis is a compulsory subject of the BGKU Bachelor's programme in Mathematics. The course aims to provide first-year students with systematic knowledge of the basics of classical analysis of univariate real functions. Educational activities (teaching and organisation of students' self-study) are aimed at students to master the classical methods of mathematical analysis, theoretical principles, and basic applications of mathematical analysis in various problems of mathematics, mechanics, other subject areas, their use in further courses in mathematics and mechanics. It is also necessary to promote the development of critical and logical thinking of students, research skills, and instrumental competencies.

In this example, we describe the learning of mathematics students only. Their groups in BGKU are small: each study year there are 8 to 12 students. These are secondary school graduates who enter the university based on the results of an external independent evaluation.[2]

The entrance assessment of knowledge and skills of freshmen, which we conduct annually on the first days of their studies at the university, traditionally (unfortunately) reveals significant gaps in most students' basic mathematical preparation—contradictory or misinterpreted concepts, fragmented and useless knowledge. A month and a half after the start of classes, we survey first-year students to identify the difficulties they encounter in studying Mathematical Analysis. We offer students a questionnaire in which they choose one or more of the proposed reasons for difficulties. Figure 18.5 shows the histogram of the distribution of responses of students in 2018 (12 students) and 2019 (9 students). The analysis of the survey results showed that the main obstacles to the successful learning of Mathematical Analysis according to students are gaps in school basic mathematical training, in particular: inability to prove

[2]External Independent Assessment in Mathematics (EIA) is an all-Ukrainian measurement of learning outcomes in mathematics for students aged 16-17 who are completing general secondary education and plan to continue their education in higher education institutions (HEIs). The tasks for EIA are developed annually by the Ukrainian Centre for Educational Quality Assessment, kept secret until the day of the event, the same for all students, and presented in the form of a test. The results of EIA of graduates of the system of complete general secondary education are used for admission to HEIs. Only those entrants who have passed the "threshold score" can enter HEI. All entrants, whose results are above the "threshold score," will receive a score on a scale of 100-200 points and will have the right to participate in the competitive selection for admission.

theorems, inability to substantiate the use of formulas; inability to independently process the material in the textbook. In 2018, students most frequently indicated as a cause of difficulties “gaps in school mathematical preparation” (statement 3), students in 2019 predominantly chose “it is difficult for me to prove theorems and formulas” (statement 5). This result, in our opinion, shows the lack of students’ understanding of the essence of mathematics as a process of proof and the relationship between different concepts.



(1) No difficulties arise
(2) There is something I don’t understand due to the high level of abstraction
(3) I find gaps in school mathematical preparation
(4) I understand the teacher’s explanation, but very rarely can I form a hypothesis on my own, identify the essential features of a new concept, give examples and counter-examples, guess the idea of solving or proving
(5) It is difficult for me to prove theorems and formulas since it was never done at school, we only were asked to learn the formulations
(6) It is difficult to independently (without additional explanation) process the educational material in the textbook
(7) I find it difficult to remember the definitions and formulations of the theorems
(8) Not enough patience / no habit to do homework

FIGURE 18.5. Survey results of first-year students majoring in Mathematics.

Also, in addition to closing gaps in school mathematics knowledge, our goal was to give students self-confidence, direct experience of mathematical discovery, and the joy and satisfaction of their mathematical research. To achieve this goal and implement the objectives of the course, the entire educational process in the Mathematical Analysis course was based on research-oriented approaches to learning (i.e., on IBME). Because the course is taught to first-year students who do not yet have sufficient experience in guiding the trajectory of their learning, structured and guided inquiry prevail. The sequence of involving students in active learning can be described by a chain: motivation (raising interest) $\to$ active action under the guidance of a teacher (constructing new knowledge) $\to$ own initiative (independently stating problems, proposing alternative solutions, and so on).

The student’s learning and cognitive motivation depend on whether the learning goals become a motivated need and personal value and interest for them, and to what extent the educational material meets these needs, values, and interests. Learning methodology largely takes into account the students' attitude to their educational activities. To form a positive permanent motivation for learning, it is important that each student feels like a subject of the educational process in which s/he plays an active role, consciously striving for self-improvement.

The specific motivational background is created by the mathematical content itself, which has several features. Such features include, in particular: a high level of abstraction; complex logical structure of many definitions and theorems; the orientation of the content is not so much on the assimilation of specific information, as on the mastery of a certain mode of action; dialectical interaction of strict proofs and heuristic considerations; the key role of tasks that motivate research activities; significant internal connections between different topics; wide possibilities of applications in various fields; as well as maximum accuracy and persuasiveness, creative inexhaustibility, beauty, and aesthetic perfection. We try to use the motivational potential generated by these features not only to stimulate the situational activity of students but also to form in them a deep inner interest in mathematics. To this end, real-life problems, mathematical problems that challenge thinking, are proposed. For example, at the beginning of the study of the topic "Definite Integration" the teacher offers students several practical problems that lead to the integral (see Section 8.4). Furthermore, acquaintance with numerical series begins with the search for "Achilles' heel" in Zeno's paradox about Achilles and the tortoise. This introduction aims to attract students, arouse interest and enthusiasm, which will give enough impetus, help to further master complex, abstract, and even boring, but necessary things and see in them a kind of beauty and harmony, as well as enjoy the mathematical activities.

Familiarity with the concept of the limit, the operation of boundary-crossing causes considerable difficulties for first-year students, including psychological, because it is something fundamentally different from what they learned in secondary school. Therefore, special attention is paid to the formation of mathematical concepts based on conceptual understanding.

To stimulate students' active mental activity, a teacher encourages students to use earlier learned material in their considerations to explain a new idea, provoking the formulation of inquiry questions that help students draw conclusions, encourage and support discussion, reflection, mutual assistance, and mutual learning. For example, to bring first-year students to the concept of continuity at a point x_0, the teacher shows on the screen 5 to 7 graphs of functions, of which only one is continuous at x_0, and invites students to find the extra one. Students do it easily. But it is difficult for them to explain their choice in mathematical terms. Here, as a rule, discussions start, which ultimately lead to the idea of using the concept of a limit (studied earlier) to define continuity. Continuing this 'game', students come to the concept of discontinuity points and their classification.

Learning is a social activity, it is what we do together in interaction with each other. Therefore, we were interested in how to organise this joint activity in and out of class, in what is the role of the teacher in this activity, and in how to ensure effective pedagogical mediation. The main goal is to achieve an improved conceptual understanding of mathematics by students and to achieve their cognitive development in general through the joint construction of knowledge in the so-called "Zone of Proximal Development" (Vygotsky, 1978, 1987). In the next section, examples from our practice illustrate attempts to achieve this goal, some positive results, and also some problems.

18.2.3. Description of the Lesson. The lesson duration was 80 minutes; the class was attended by 8 students; the topic was: absolute and conditional convergence of a numerical series. The purpose is to acquaint students with the concept of absolute (conditional) convergence of a series and the possibilities of applying the convergence of non-positive series to research, to form a conceptual understanding of these concepts, and to improve research and procedural skills.

Expected learning outcomes are: *Knowledge*—the concept of absolutely and conditionally convergent series, a sufficient condition for the convergence of a non-positive series; *Skills*—the study of absolute (conditional) convergence of series; and *Research and procedural skills*—the ability to make empirical reasoning, make assumptions, and understand the essence of mathematical proof; the ability to present one's judgements. Requested preliminary knowledge consists of the concept of a numerical series, its convergences/divergence, its sum; properties of convergent series; a necessary condition for the convergence of the series; Cauchy convergence criterion; convergence tests of positive series; Leibniz criterion of convergence of alternating series; understanding what a sufficient and necessary condition is; understanding in which cases the use of the necessary condition of convergence of a series can be effective and the ability to use it; skills of investigating of convergence of positive and alternating series. The lecture was conducted in the form of a video conference on the Zoom platform (due to the COVID-19 pandemic) with PLATINUM participants from different partner universities present.

The lecturer brought students to the definition of absolute and conditionally convergent series gradually. First, she posed the following problem:

Investigate the convergence of the non-positive series $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$.

This created the conditions in which the student must recognise (see or feel) the need for new knowledge. Note that by choosing this series for research, the teacher anticipated and even deliberately provoked students to a misconception, which in turn (after being rejected) intensified the intrigue and desire to solve the problem. Below is shown how a way to solve the problem was found (excerpt of the discussion).

Excerpt

S1: (*immediately*) The series is convergent based on the comparison test $\frac{\sin(n)}{n^2} \leq \frac{1}{n^2}$, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, so this series is convergent.

(Pause)

Lecturer (L): Does everyone agree with *S1*?

S2: No.

L: Why?

S2: Because the comparison test is for positive series, and our series contains both positive and negative terms.

L: Maybe based on the Leibniz criterion?

S3: No, it is not possible, because the signs of the terms of the series do not alternate: the first three terms are positive, then a few negative, then again positive, and so on.

(Pause)

L: Okay. Let's form from this series an alternating series:

$$\left(\frac{\sin 1}{1^2} + \frac{\sin 2}{2^2} + \frac{\sin 3}{3^2}\right) + \left(\frac{\sin 4}{4^2} + \frac{\sin 5}{5^2} + \frac{\sin 6}{6^2}\right) + \left(\frac{\sin 7}{7^2} + \frac{\sin 8}{8^2} + \frac{\sin 9}{9^2}\right) + \cdots (*)$$

In it, the first term is positive, the second is negative, the third is positive again, etc. We will separately collect successive positive and negative terms in groups.

S4: It will not help. Because according to the Leibniz criterion, the absolute values of the terms of a series must decrease and tend to zero, and in the series, you have formed, it is unknown whether this is the case.

L: At least, it's not obvious. Well, if we somehow proved that the absolute values of the terms of the formed series decrease and tend to zero, then could it be concluded that the series under study is convergent?

S5: Yes.

S3: No. Because, even if we established that this series is convergent, it will still not follow the convergence of the initial series. The series (*) is formed by grouping the terms of the given in the problem series (combining in parentheses), without changing the order of their sequence. We know that a convergent series has the associative property. That is, if we have a *convergent* series and group its terms, we also get a convergent series. Not the other way around. The other way around is even wrong, which can be easily illustrated with an example.

L: Convincing. And what other tools are there?

S6: A necessary condition for convergence. But it also does not help to solve the problem, because $\frac{\sin n}{n^2} \to 0$ if $n \to \infty$.

S7: And there is the Cauchy criterion and the definition of the convergence of the series. But they are inconvenient for practical use.

L: So what do we do?

S1: (*emotionally*) But our series is still convergent! Well, look: if you take a positive series $\sum_{n=1}^{\infty} \frac{|\sin n|}{n^2}$, it is convergent, based on comparison test. And in our series, some terms are just negative. Well, for example, take the sum $1+2+3+4+5 = 15$. And now we will change some terms for the opposite, the sum will decrease. For example, $1 - 2 + 3 + 4 + 5 = 11 < 15$. It will be the same in the case of a series.

L: But we know that it is risky to automatically transfer facts that are valid in finite sets to infinite sets.

S1: Well, that's just a hypothesis

L: Then formulate it.

S1: (*formulates a hypothesis*) If the series $\sum_{n=1}^{\infty} |a_n|$ is convergent, then the series $\sum_{n=1}^{\infty} a_n$ is convergent.

The given excerpt is an example of using the inquiry approach in the organisation of research activity of students. With a series of purposeful inquiry questions, the teacher directed the students' progress to their independent formulation of the hypothesis. Despite the lack of direct contact among the audience, an atmosphere of cooperation was created (although not without certain losses), and the teacher periodically moved from the role of facilitator to the role of the team member, offering options for solving the problem. Changing this role had a purpose—not to direct students to the shortcut "a straight line"—but to lead them through a 'maze' with access to all sorts of 'dead ends.' This technique contributed to the conceptual understanding of the research problem and helped to develop systemic and flexible thinking.

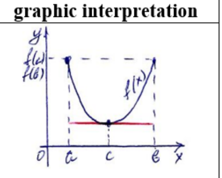
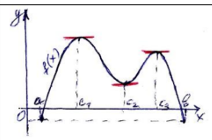
One of the indicators of conceptual knowledge is the ability to apply it in practice. In addition to conceptual understanding, procedural knowledge is very important here. Its formation, as well as the deepening of conceptual knowledge 'in action,' occurs mainly in practical classes and in the process of solving independent practice problems. Our practice has shown that a significant problem for students is the ability to recognise a particular mathematical theory in a practical problem, the content of which does not directly indicate this theory. Thus, in one of the practical classes on "Numerical Series," students were given the task to find the area and perimeter of the Koch Snowflake. But they could not independently 'see' a numerical series in this problem. Instead, the same students, in the same class, brilliantly coped with a rather difficult (conceptual) task to study the convergence of a series, and they solved it in different ways.

18.2.4. Organisation of Extracurricular Collective Work of Students. In Section 18.2.3 we described how it is convenient to organise the collective work of students in the classroom, real or virtually (although less successfully). And how to organise the interaction of students when doing homework. We tried to find models for organising such cooperation outside of classes and tested three forms: the so-called 'conceptual tables,' and the FORUM and WIKI tools in the LMS MOODLE in an e-learning course (ELC), which were developed by lecturer Maria Astafieva and used in the educational process.

A conceptual table is the summarised, organised, and structured information about the content on a particular topic. The table is filled by a group of students (2-4 people) in class, and more often in extracurricular time. They formulate questions themselves and answer them. At the same time, students demonstrate an understanding of the essence of concepts, facts of this topic, their connection with the previously studied material, the ability to correlate different forms of presentation of a mathematical topic (verbal, symbolic, graphic). The organisation of the next activity with the filled conceptual tables depends on what purpose the teacher pursues: to continue training or to check and estimate knowledge. Depending on this, group discussions can be employed, mutual reviews or a check of the table by the teacher. An example of a conceptual table on Rolle's Theorem is given below (Table 18.1).

Conceptual table

Course: Mathematical analysis. Topic: Rolle's theorem
Students: S1; S2; S3. Date: December 19, 2019

Question	Answer		
	verbal	symbolic	graphic interpretation
How is Rolle's theorem formulated?	If a real-valued function is continuous on a proper closed interval and differentiable at each of its interior points, and at the ends of the interval it acquires equal values, then inside the interval there is a point at which the derivative of the function is equal to zero.	$f(x)\in C[a;b]\wedge$ $\wedge\exists f'(x)\forall x\in(a;b)\wedge$ $\wedge f(a)=f(b)\Rightarrow\exists c\in(a;b)$ $f'(c)=0$	
Is point c unique?	No. For example, the function $\sin x$ on the proper closed interval [0; 2π] satisfies the conditions of Rolle's theorem and inside this interval there are two points $\pi/2$ and $3\pi/2$, at which the derivative of the sinus ($\cos x$) is equal to zero.	$\exists c_1,c_2,c_3\in(a;b):$ $f'(c_1)=f'(c_2)=f'(c_3)=0$	
Is it possible to find a point (points) c by using the theorem?	The theorem does not answer the question of where exactly on the interval is the point c; it only guarantees the existence of such a point.	-	-
What conditions for the existence of a zero derivative are given by the theorem: sufficient; necessary; necessary and sufficient?	The theorem sets sufficient conditions	If $A\Rightarrow B$, then A is sufficient for B	-

Example of a conceptual table (*continued on next page*).

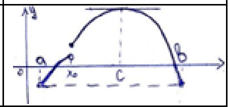
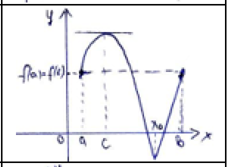
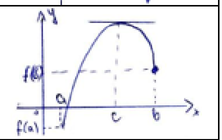
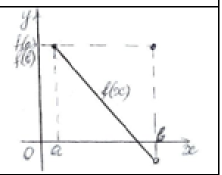
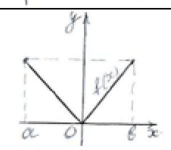
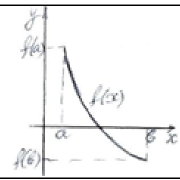
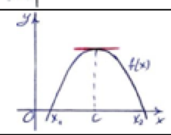
Are the conditions of the theorem necessary?	No. None of the three conditions is necessary. That is, the condition is not fulfilled, and the zero derivative exists. This is well illustrated by the appropriate graphics for the cases:	-	-
	a) the function is not continuous on the proper closed interval, and the other two conditions are satisfied	$\exists c \in (a;b): f'(c)=0$, although x_0 – the breakpoint of the function	
	b) inside the proper closed interval there are points at which the function is not differentiable, and the other two conditions are fulfilled	$x_0 \in (a;b) \wedge \nexists f'(x_0)$, but $\exists c \in (a;b): f'(c)=0$.	
	c) at the ends of the proper closed interval the values of the function are not the same, and the other two conditions are satisfied	$f(a) \neq f(b)$, but $\exists c \in (a;b): f'(c)=0$.	
Are the conditions of the theorem significant?	Yes. Each condition is significant and if it is not fulfilled, there may be no zero derivative (see relevant graphics):	-	-
	a) the condition of continuity of the function is not fulfilled	b – the breakpoint of the function and $f'(x) \neq 0\ \forall x \in (a;b)$	
	b) the condition of differentiation is not fulfilled	$\nexists f'(0), 0 \in (a;b)$ i $\nexists c \in (a;b): f'(c)=0$.	
	c) at the ends of the interval the values of the function are not the same	$f(a) \neq f(b)$ i $\nexists c \in (a;b): f'(c)=0$.	
What is the consequence of Rolle's theorem?	For example: "Between two zeros of a differentiable function is a zero of its derivative"	$f(x_1)=f(x_2)=0 \wedge$ $\wedge \exists f'(x) \forall x \in (x_1;x_2) \Rightarrow$ $\Rightarrow \exists c \in (x_1;x_2): f'(c)=0$	

Table 18.1. Example of a conceptual table on Rolle's Theorem.

Another activity that we used in Mathematical Analysis ELC in the LMS Moodle for students to perform tasks together is the Wiki activity. This tool allows participants to add and edit web pages. The history of all changes is preserved and this allows the teacher to follow the trajectory of each student and respond on time and evaluate the educational process, using the technique of formative assessment. The Wiki collections created by students are a virtual analogue of the conceptual table (Astafieva et al., 2019).

The initiative is an important component of active learning, and at the same time, evidence of a high level of student motivation, and active involvement in the research process is expressed in the ability to inquire, to put forward new ideas or proposals for research, to offer different solutions, and to formulate new tasks.

An indicator of the attainment of a high level of conceptual knowledge by some students is the discussion of the topic "Numerical Series" at the Forum of the electronic training course "Mathematical Analysis." At the end of the study of the topic (in May, 2020), the teacher invited freshmen to ask mathematical questions about this topic on the forum and give answers to them. Five students asked conceptual questions and participated in the discussion. Here is an example of one of these questions and the discussion of students caused by it.

Excerpt

S1: The Leibniz criterion of convergence of an alternating series requires that the sequence of its terms decreases and tends to zero. How important is the condition of decreasing?

S2: The condition of decreasing the sequence of terms of the series is significant. If this sequence converges to zero but does not decrease, then the series may be divergent. An example of such a series is

$$\frac{1}{2}-\frac{1}{2^2}+\frac{1}{3}-\frac{1}{3^2}+\frac{1}{4}-\frac{1}{4^2}+\frac{1}{5}-\frac{1}{5^2}+\cdots \qquad (**)$$

The terms of this series tend to zero, but non-monotonically: $a_1 > a_2$, $a_2 < a_3$, $a_3 > a_4$, $a_4 < a_5$, etc.

Let us show that this series is divergent. To do this, let us group its terms as follows:

$$\left(\frac{1}{2}-\frac{1}{2^2}\right)+\left(\frac{1}{3}-\frac{1}{3^2}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n^2}\right)+\cdots=\sum_{n=2}^{\infty}\left(\frac{1}{n}-\frac{1}{n^2}\right)=\sum_{n=2}^{\infty}\frac{n-1}{n^2}$$

We obtained the divergent series, it is compared with a harmonic series $\sum\limits_{n=1}^{\infty}\frac{1}{n}$.

Thus, the series (**) is also divergent, because if it was convergent, then the series formed from it by grouping the terms would be convergent. The significance of the condition of decreasing the sequence of terms of a series in the Leibniz criterion is proved.

S3: The condition of decreasing the sequence of terms of the series is important. The following example proves this. Let's take the series $1-\frac{1}{2^2}+\frac{1}{3}-\frac{1}{4^2}+\frac{1}{5}-\frac{1}{6^2}+\cdots$.

The terms of this series tend to zero, but they decrease, then increase. Let's form the series

$$\left(1-\frac{1}{2^2}\right)+\left(\frac{1}{3}-\frac{1}{4^2}\right)+\left(\frac{1}{5}-\frac{1}{6^2}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{(n+1)^2}\right)+\cdots$$

The n-th term of this series is $a_n=\frac{1}{n}-\frac{1}{(n+1)^2}=\frac{(n+1)^2-n}{n(n+1)^2}=\frac{n^2+n+1}{n(n+1)^2}$.

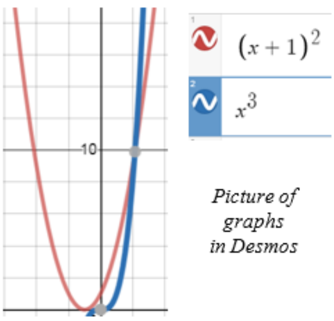
The series is divergent, because we have $\lim\limits_{n\to\infty}\left(\frac{n^2+n+1}{n(n+1)^2}:\frac{1}{n}\right)=1$, based on the comparison test, and the series $\sum\limits_{n=1}^{\infty}\frac{1}{n}$ is divergent. Therefore, the initial series is divergent, because if it was convergent, then after combining the terms in parentheses, it would remain convergent.

S4: Although the condition of decreasing the sequence an is essential in the Leibniz criterion, it is not necessary. Take, for example, the following series:

$$\frac{1}{2^2}-\frac{1}{2^3}+\frac{1}{3^2}-\frac{1}{3^3}+\frac{1}{4^2}-\frac{1}{4^3}+\frac{1}{5^2}-\frac{1}{5^3}+\cdots$$

The sequence of its terms $\{a_n\}: \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{3^2}, \frac{1}{3^3}, \frac{1}{4^2}, \frac{1}{4^3}, \frac{1}{5^2}, \frac{1}{5^3}, \cdots$ converges to zero. But this is not a decreasing sequence, because in it the first four terms are decreasing, and the fifth is greater than the fourth; the sixth member is less than the fifth, and the seventh is greater than the sixth; etc. Nevertheless, the series is convergent, because it can be interpreted as the sum of convergent series $\sum_{n=2}^{\infty} \frac{1}{n^2}$ and $\sum_{n=2}^{\infty} \frac{-1}{n^3}$.

S5: A good example was provided by S4. But I don't understand how it can be seen that the sequence $\{a_n\}$ is not decreasing. The phrase "etc." did not convince me. There was no certainty that starting with the fifth term, as *S4* claimed, each term with an odd number is greater than the previous one, i.e., $\frac{1}{(n+1)^2} > \frac{1}{n^3}$. But it turns out that this is indeed the case. The graphic image in Desmos[3] helped me to see it.



Picture of graphs in Desmos

$$(n+1)^2 < n^3 \qquad \forall n > 2.$$

Then:

$$\frac{1}{(n+1)^2} > \frac{1}{n^3} \qquad \forall n > 2$$

Thus, we have:

$$\frac{1}{n^3} < \frac{1}{n^2} \qquad \forall n > 1$$

It is obvious. But already $\frac{1}{4^2} > \frac{1}{3^3}$, $\frac{1}{5^2} > \frac{1}{4^3}$, $\frac{1}{6^2} > \frac{1}{5^3}$, etc.

That is the sequence $\{a_n\}$ is not monotonic.

The fragment above shows that the students who participated in the discussion had the ability to:

- independently formulate a research problem and willingness to work on its solution;
- find the right idea to solve the problem;
- understand what is sufficient, necessary, and essential condition;
- understand the essence and methods of mathematical proof;
- feel the internal need for full evidence;
- make strict logical reasoning;
- choose convincing arguments for argumentation and critically evaluate provided arguments;
- apply previous experience and knowledge to solve a new problem; and to
- establish a connection between different interpretations of mathematical concepts and facts, in particular, to use a graphic image for illustration and argumentation.

18.3. Evaluating Effectiveness of IBME to Achieve Educational Goals

Evaluation of the effectiveness of IBME to achieve educational goals was carried out according to a scheme developed by the BGKU team based on a template created by project participants from the Complutense University of Madrid (see Chapter 9),

[3] www.desmos.com

and taking into account student feedback, teacher self-analysis and collective discussions of project participants in academia. The evaluation consists of five blocks.

The First Block is the general information about the lesson: date, course, speciality, course, number of students, topic and purpose of the lesson, type of lesson (lecture, practical), duration of the lesson, equipment, software used during the lesson, expected learning outcomes, prior knowledge that students should have.

The Second Block is a description of educational activities during the lesson: the actions of teachers and students.

The Third Block is the characteristics of the student group (formulated by the teacher), which allows determining the level of internal motivation to study mathematics and what it is caused by; whether students have experience in managing their learning trajectory; the initiative of students in self-study; the ability to persistently, purposefully overcome the difficulties and obstacles that arise in the process of solving a problem.

The Fourth Block is a description of students' activities during the lesson: the level of involvement in the educational process; how actively students participate in the study (discussion of problematic issues); ability to formulate different types of questions: clarifying, research, hypothetical; research and procedural skills demonstrated by students during the class; ability to put forward their hypotheses; ability to self-reflection (what I learned, how my knowledge, skills, abilities have changed); how the students saw (felt) the relationship with the previously studied material.

The Fifth Block of assessment of a lesson provides an assessment (low, average, high) of achievement of the purposes set by the teacher and its substantiation.

According to this scheme, we present the analysis and evaluation of the lesson "Absolute and conditional convergence of a numerical series," described in Section 18.2.3.

The First Block. General information about the lesson (already presented in Section 18.2.3)

The Second Block. Description of educational activities during the lesson.

The teacher implemented the 5E model of instruction (Bybee et al., 2006) during the lecture.

In the *Engage* phase, students were asked to investigate the convergence of a series $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$. Because the day before, when studying positive series, a seemingly similar positive series $\sum_{n=1}^{\infty} \frac{\sin(\pi/n)}{n^2}$ was investigated for convergence, the teacher thus 'programmed' an error—due to external similarity to consider the proposed series as positive. This plan worked and some students decided that the series is positive and one of the signs of convergence of the positive series can be used directly to investigate it. However, the series is not positive. Students noticed this upon closer analysis and were therefore convinced that the comparison test did not work. There was a need to look for a new one.

During the *Explore* phase, students tried to find a way to solve the given problem. At this stage, the teacher played the role of facilitator or member of the team of 'researchers' when discussing the problem and ways to solve it: the main tasks are then to help, guide, provoke, and ask questions that push to formulate a hypothesis. During the discussion, students analysed, reflected, asked questions that help to advance in the study, expressed ideas and opposed; formulated a hypothesis and looked for a way to prove it.

In the *Explain* phase, students together with the teacher implemented the idea of proof found during the previous phase. The teacher wrote down the proof of the theorem, involving students in commenting. Students together with the teacher also wrote down the proof. The teacher formulated the definition of absolutely and conditionally convergent series.

In the *Elaborate* phase, the teacher sought to expand the conceptual understanding of the proven sign of the convergence of the series. To do this, he enquired: "We have just found that a series is convergent if the series formed from the absolute values of its terms is convergent. And what can be said about the convergence of this series, if the series of absolute values of its terms is divergent?" Based on the discussion, students concluded that a proven condition is not necessary. They gave a suitable example.

Several specially selected exercises were solved next that helped to notice one important detail: if the divergence of a series $\sum_{n=1}^{\infty} |a_n|$ is established based on D'Alembert or Cauchy criterion, then the series $\sum_{n=1}^{\infty} a_n$ is divergent. The teacher again encouraged the students to study with the question: "Do you think this is a coincidence?" In the process of a short discussion led by a teacher, a reasonable answer to the question was given. The teacher together with the students concluded how the established fact can help in practice.

In the *Evaluate* phase students summed up, upon the request of the teacher, what they have learned and what is the practical value of the knowledge gained.

Block 3. Characteristics of the student group.

Students demonstrated intrinsic motivation to study mathematics, had little experience in managing the trajectory of their learning; a large part of the group could persistently, purposefully overcome the difficulties and obstacles that arose in the process of solving the problem (for example, if the next step is not obvious or it is necessary to restore some previous knowledge for further progress).

Block 4. Characteristics of students activities during the lesson.

During the lesson, students were involved in the learning process. The teacher created conditions in which students had to recognise the need for new knowledge, because they found themselves in a situation where the knowledge they already was not enough to solve the problem.

In the process of research, most students actively participated in the discussion, formulated questions independently. Some students showed intrinsic motivation to solve the problem without the support of the teacher. At the end of the lesson, students assessed their progress and expressed their impressions of the lecture in a chat.

Block 5. Evaluation of the achievement of goals.

The purpose of the lecture was achieved. In particular, students under the guidance of a teacher concluded that the absolute convergence of a series was a sufficient condition for its convergence, and proved it. The examples demonstrated the ability to apply the proven criterion to the investigation of the convergence of non-positive series. In addition, during the class, students demonstrated an understanding of the relationships with previously studied material. Such a result indicates the effectiveness of IBME: selected educational content, the organisation of active research activities of students, learning through scaffolding gave a positive result.

18.4. Discussion of the Case in the Community of Inquiry

Throughout the semester, Mathematical Analysis classes were attended by members of the academic community – PLATINUM project participants and other interested teachers. The described case has been repeatedly discussed in the community. The ways and methods of application of IBME used by the teacher, their expediency, and efficiency were discussed. The problems faced by teachers and students (what worked and what didn't? why?) were considered, especially during the implementation of IBME in distance and blended learning (through 2020 during quarantine), and recommendations for their solution were developed.

The judgements of the students, which they expressed about their academic achievements, attitudes to mathematics, and the teaching methods used during Mathematical Analysis were also taken into account. Students noted the positive dynamics of the achieved results in terms of subject knowledge and skills. In addition, they indicated a significant improvement in understanding mathematical facts, the acquisition of certain research skills (ability to observe, analyse, doubt, the ability to ask right questions, reason logically, express hypothesis, test it, prove facts, properly express opinions, draw conclusions and generalisations, etc.). They also noted the development of imagination, increased interest, and motivation, the ability to learn independently. Students responded positively to the teaching methods used (comfortable, friendly atmosphere of discussion of problems, ideas, motivation to search and research, learning to ask the right questions, help, etc.). And the fact that the discussion (questions, answers, discussions, reflections) on the forum of the distance e-course "Mathematical Analysis" continued even after the students passed the exam, is evidence of their persistent interest, intrinsic motivation, and comfort in learning. There are, of course, some unresolved issues, in particular, there are difficulties with the processing of book texts. Below are some excerpts from students' considerations.

> My understanding of mathematics has greatly improved. Now I not only understand the proof but also draw the right logical conclusions, ask the right questions to move forward. And it's very interesting. I liked mathematics back in school, but now I felt what a beautiful and interesting science it is, I loved it.
>
> It has become much easier to study Mathematical Analysis than it was at the beginning. Although even now there are difficulties – I do not always understand everything from the first time. But I consider it great progress that I already know how to ask competently, to explain what I do not understand. I think this is the main thing I learned in half a year.
>
> During the six months that we have been studying Mathematical Analysis, it is thanks to the teaching methods that my perception of mathematics has changed a lot. I began to see and understand the connections between different mathematical concepts and facts, even from different mathematical courses.
>
> I learned to use mathematical symbols. Thanks to geometric interpretations I intuitively feel some mathematical facts, ideas of their proof. But there are still problems: it is difficult to study the material in the textbook, I do not always understand the evidence written there.
>
> I became more confident. I'm not afraid to express my opinion, to suggest the idea of proving a theorem or solving a problem.
>
> The teacher always helps when needed, but never gives a ready-made solution or answer, we always come to them ourselves. The big problem was the inability to read mathematical literature on my own, even a textbook. It is getting much better now.
>
> Discussing tasks in small groups helps a lot to understand the learning material. After all, each of us is faced with a problematic issue, to solve which everyone needs to express

their opinion. But the discussion is much more effective when the teacher participates in it. He corrects the course of our discussion, helps to resolve disputes, gives some clarifications.

I'm used to proving every theorem now, we didn't practice that at school. I distinguish between necessary and sufficient conditions. I learned to understand what it is about, you need to ask the right question, it helps in understanding the material. There is no longer any fear of making assumptions or hypotheses. It remained difficult for me: not to confuse something in the definition of the limit in the "$\epsilon - \delta$" language, but I almost overcame it.

At the end of the academic year, I started to have my hypotheses to solve the given problem. The teacher always encourages me to ask the right questions, give examples and counterexamples. Now I'm set up to prove the problem myself, not just rewrite and learn by heart like it was done at school.

The best results of the exam in the Mathematical Analysis course of entrants 2019 (the teacher modified the course based on IBME), compared to 2018 also confirm the effectiveness of the used IBME strategy and justify the pedagogical expectations (see Figure 18.6).

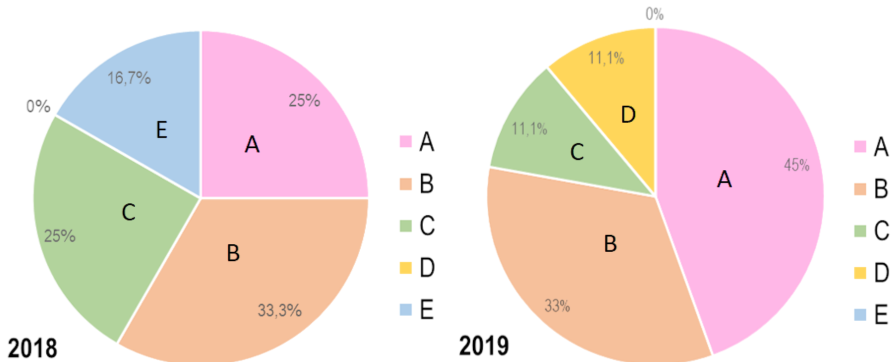


FIGURE 18.6. Mathematical analysis of exam results (12 students in 2018, 9 students in 2019) (coloured version in the ebook).

As it was mentioned above, the need for transition to blended and distance learning, which is urgent today, creates certain problems in the organisation of the educational process. The main problems of distance learning, which were noted by students, are low quality of communication during video conferencing, lack of constant communication between students and the teacher, which makes it harder to understand the learning material, and learning becomes slower. Students unanimously preferred in-person learning because of the possibility of direct communication and cooperation. Among the advantages of online classes, students included only the fact that after the class you can watch a video at a convenient time if necessary and that you do not need to spend time on the way to university. However, the students appreciated the conducted classes in an online format, noting that the teacher put maximum effort to make the lectures and practical classes full-fledged.

The teacher also noted the worsened conditions for communication in online learning through video conferencing. In particular, the possibility of organising productive work in small groups is practically lost, because the teacher cannot hear the discussion in all groups at the same time, as it happens in in-person education. Therefore, the teacher cannot react on time, join the discussion, leaving the process of forming students' research skills to themselves.

In addition, our 2018 survey of students on their perception of ELC teaching materials in mathematical courses, posted in the LMS Moodle, shows that most ELCs did not provide interactive learning and did not create positive intrinsic motivation in students, i.e., did not promote active, research-oriented learning in partnership. We offered some didactic and methodological approaches to the preparation of content and organisation of activities in ELC in Mathematics during the implementation of blended learning based via LMS Moodle to improve their quality and efficiency (Astafieva et al., 2019).

It is important to note that the practice of using IBME is of interest to teachers of mathematics courses, who are not participants of the project but are part of the academic community. Dissemination of community outcomes occurs through the exchange of experiences with colleagues in seminars and the involvement of colleagues in research, which is reflected in joint publications, the project website, and social media pages.

We are aware that our practice is neither the only correct one nor the only possible to achieve high results in learning mathematics. The described case only confirms that inquiry-based approaches can be effective, that our proposed approach to learning with its help can be useful, and that some of the ideas about IBME can be implemented in all practices of mathematics teachers.

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