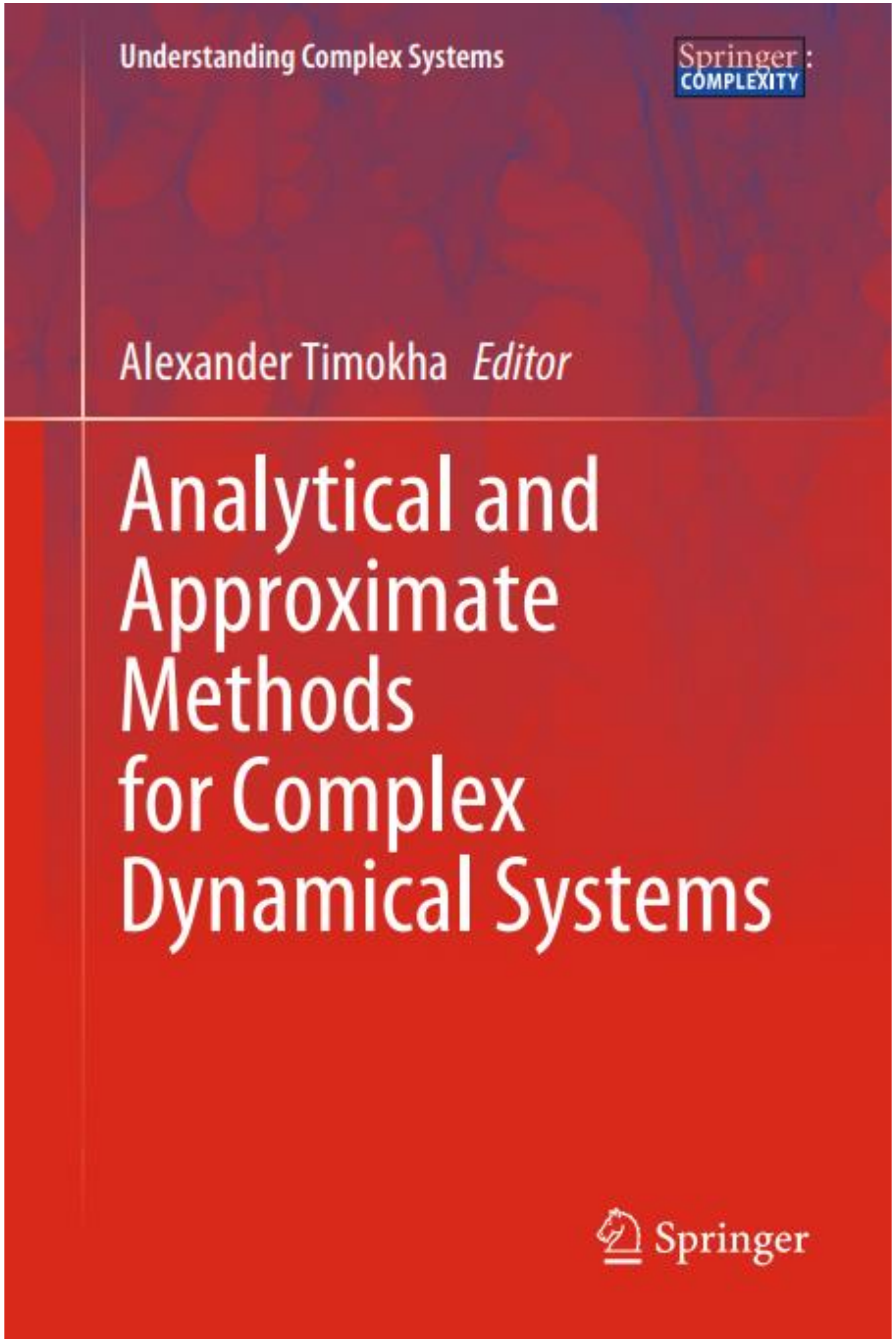

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Contents

On Group Classification of Nonlinear Heat Equation: Algebraic Approach

Sofiia Huraka and Oleksandra Lokaziuk

Abstract Using the classical Lie theorem on realizations of Lie algebras by vector fields on the line, we substantially simplify the proof of the known results on group classification of the class of (1+1)-dimensional nonlinear evolution equations $u_t = H(u_{xx})$.

Keywords Evolution equations · Group classification · Equivalence transformation · Symmetry · Algebraic approach

1 Preliminary Analysis

The purpose of the present paper is to clarify and simplify the proof of results of Akhatov, Gazizov and Ibragimov [1, Sect. 4] on the group classification of the class of evolution equations

$$u_t = H(u_{xx}), \tag{1}$$

where $u_t = \frac{\partial u}{\partial t}$ and $u_{xx} = \frac{\partial^2 u}{\partial x^2}$, and H is an arbitrary smooth function of u_{xx}. If H is linear, Eq. (1) is known as linear heat equation, and, therefore, the class (1) is sometimes called the class of nonlinear heat equations.

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Let us redefine $u_0 := u_t$, $u_1 := u_x$, $u_{11} := u_{xx}, \dots$ and, therefore, Eq. (1) is rewritten in the form

$$u_0 = H(u_{11}). \tag{2}$$

Throughout the paper, by default the indices $i, j, \dots = 0, 1$, and we are assuming summation over the repeated indices. Functions with subscripts denote derivatives with respect to the corresponding variables. Excluding from consideration the case of the linear heat equation $u_0 = u_{11}$, let us suppose that H is the nonlinear function of the variable u_{11}. We refer interested readers to the recent work by Koval and Popovych [9], where some bugs in the group analysis of the linear heat equation are corrected. Below, we use the following notation for vector fields from invariance algebras of equations from the class (2)

$$Q^1 = \partial_t; \quad Q^2 = \partial_x; \quad Q^3 = \partial_u; \quad Q^4 = 2t\partial_t + x\partial_x + 2u\partial_u; \quad Q^5 = x\partial_u,$$
$$Q^{6a} = 2t\partial_t - x^2\partial_u; \quad Q^{6b} = x\partial_x - 2t\partial_u; \quad Q^{6c} = (1-p)t\partial_t + u\partial_u, \ \ p \neq 0, \pm\tfrac{1}{3}, 1,$$
$$Q^{6c_1} = 2t\partial_t + 3u\partial_u; \quad Q^{6c_2} = 4t\partial_t + 3u\partial_u; \quad Q^{7c_1} = u\partial_x; \quad Q^{7c_2} = x^2\partial_x + xu\partial_u,$$

where Q^{6c_1} and Q^{6c_2} are partial cases of the operator Q^{6c} for $p = \frac{1}{3}$ and $p = -\frac{1}{3}$, respectively.

Theorem 1 (Equivalence group, see [1, Sect. 3.3]) *The equivalence group $G^{\sim}$ of the class (2) is generated by the transformations*

$$\tilde{t} = \mu^0 t + \mu^1; \quad \tilde{x} = \nu^0 x + \nu^1,$$
$$\tilde{u} = \kappa^0 u + \kappa^1 x^2 + \kappa^2 x + \kappa^3 t + \kappa^4; \quad \tilde{H} = \frac{\kappa^0}{\mu^0} H + \kappa^3,$$

where $\mu^0, \mu^1, \nu^0, \nu^1, \kappa^0, \dots, \kappa^4$ are arbitrary constants such that $\mu^0\nu^0\kappa^0 \neq 0$.

The vector field [1, Sect. 4]

$$Q = \xi^0(t, x, u)\partial_t + \xi^1(t, x, u)\partial_x + \eta(t, x, u)\partial_u \tag{3}$$

belongs to the maximal Lie invariance algebra $\mathfrak{g}_H$ of an equation from the class (2) for any H if and only if it satisfies the following classifying equation

$$(\zeta^0 - \zeta^{11}H')\big|_{u_0 = H(u_{11})} = 0, \tag{4}$$

where ζ^0 and ζ^{11} are the corresponding coefficients of the first and second prolongations of the vector field (3), $H' := \partial H(u_{11})/\partial u_{11}$.

Rewriting Eq. (4) in an expanded form leads to

$$\begin{aligned}\eta_0 + \eta_u H - H(\xi^0_0 + \xi^0_u H) - u_1(\xi^1_0 + \xi^1_u H)\\ - H'\big[\eta_{11} + 2\eta_{1u} u_1 + \eta_{uu} u_1^2 + \eta_u u_{11} - 2u_{1j}(\xi^j_1 + \xi^j_u u_1)\\ - H(\xi^0_{11} + 2\xi^0_{1u} u_1 + \xi^0_{uu} u_1^2 + \xi^0_u u_{11}) - u_1(\xi^1_{11} + 2\xi^1_{1u} u_1 + \xi^1_{uu} u_1^2 + \xi^1_u u_{11})\big] = 0.\end{aligned}$$

Splitting this equation with respect to u_{01}, $u_1 u_{01}$, u_1^3, u_1^2, u_1 and 1 yields

$$\xi^0_1 = 0; \quad \xi^0_u = 0; \quad \xi^1_{uu} = 0; \quad \eta_{uu} - 2\xi^1_{1u} = 0, \tag{5}$$

$$\xi^1_0 + \xi^1_u H + (2\eta_{1u} - \xi^1_{11} - 3\xi^1_u u_{11})H' = 0, \tag{6}$$

$$\eta_0 + (\eta_u - \xi^0_0)H - (\eta_{11} + (\eta_u - 2\xi^1_1)u_{11})H' = 0. \tag{7}$$

General solution of system (5) has the form

$$\xi^0 = \xi^0(t); \quad \xi^1 = \alpha(t,x)u + \beta(t,x); \quad \eta = \alpha_1 u^2 + \gamma(t,x)u + \delta(t,x),$$

where $\xi^0(t)$, $\alpha(t,x)$, $\beta(t,x)$, $\gamma(t,x)$, and $\delta(t,x)$ run through the set of smooth functions of their arguments. By substituting these expressions into (6) and (7), we obtain the following system

$$\alpha_0 u + \beta_0 + \alpha H + (4\alpha_{11} u + 2\gamma_1 - (\alpha_{11} u + \beta_{11}) - 3\alpha u_{11})H' = 0,$$

$$\begin{aligned}\alpha_{01} u^2 + \gamma_0 u + \delta_0 + (2\alpha_1 u + \gamma - \xi^0_0)H - [\alpha_{111} u^2 + \gamma_{11} u\\ + \delta_{11} + (2\alpha_1 u + \gamma - 2(\alpha_1 u + \beta_1))u_{11})]H' = 0.\end{aligned}$$

Splitting the above system with respect to the powers of u, we come to the system

$$\alpha_0 + 3\alpha_{11} H' = 0, \tag{8}$$

$$\beta_0 + \alpha H + (2\gamma_1 - \beta_{11} - 3\alpha u_{11})H' = 0, \tag{9}$$

$$\alpha_{01} - \alpha_{111} H' = 0, \tag{10}$$

$$\gamma_0 + 2\alpha_1 H - \gamma_{11} H' = 0, \tag{11}$$

$$\delta_0 + (\gamma - \xi^0_0)H - (\delta_{11} + (\gamma - 2\beta_1)u_{11})H' = 0. \tag{12}$$

Taking into account that $H'' \neq 0$, from Eq. (8), it follows

$$\alpha_0 = 0 \quad \text{and} \quad \alpha_{11} = 0.$$

At the same time, Eq. (10) becomes insignificant. Then, after differentiating Eq. (9) with respect to u_{11}, we get

$$-2\alpha H' + (2\gamma_1 - \beta_{11} - 3\alpha u_{11})H'' = 0.$$

Once again, we differentiate the above equation with respect to the variable x and obtain

$$-2\alpha_1 H' + (2\gamma_{11} - \beta_{111} - 3\alpha_1 u_{11})H'' = 0. \tag{13}$$

Differentiating Eq. (11) with respect to the variable u_{11} gives

$$2\alpha_1 H' - \gamma_{11} H'' = 0. \tag{14}$$

Adding Eqs. (13) and (14) deduces

$$(\gamma_{11} - \beta_{111} - 3\alpha_1 u_{11})H'' = 0.$$

Given the condition $H'' \neq 0$, we derive

$$\alpha_1 = 0; \quad \gamma_{11} = 0; \quad \beta_{111} = 0.$$

On the other hand, from (11) follows that $\gamma_0 = 0$. Then

$$\alpha = C^1 = \text{const}; \quad \beta = \beta^2(t)x^2 + \beta^1(t)x + \beta^0(t); \quad \gamma = C^2 x + C^3,$$

where $\beta^2(t)$, $\beta^1(t)$, and $\beta^0(t)$ run through the set of smooth functions of t, and $C^1, C^2, \ldots$ are arbitrary constants. Substituting these expansions into Eq. (9) yields

$$\beta^2_0 x^2 + \beta^1_0 x + \beta^0_0 + C^1 H + (2C^2 - 2\beta^2 - 3C^1 u_{11})H' = 0.$$

Splitting with respect to x yields

$$\beta^2_0 = 0; \quad \beta^1_0 = 0; \quad \beta^0_{00} = 0$$

and, therefore,

$$\beta^2 = C^4; \quad \beta^1 = C^5; \quad \beta^0 = C^6 t + C^7,$$

$$C^6 + C^1 H + (2C^2 - 2C^4 - 3C^1 u_{11})H' = 0. \tag{15}$$

Thus, the coefficients of the vector field Q become of the following form

$$\xi^0 = \xi^0(t), \tag{16}$$

$$\xi^1 = C^1 u + C^4 x^2 + C^5 x + C^6 t + C^7, \tag{17}$$

$$\eta = (C^2 x + C^3)u + \delta(t,x). \tag{18}$$

After substitution of (17) and (18) into (12), we obtain the following *classifying condition*

$$\delta_0 + (C^2x + C^3 - \xi^0_0)H - (\delta_{11} + (C^2x + C^3 - 2(2C^4x + C^5))u_{11})H' = 0. \tag{19}$$

If H is an arbitrary function, Eqs. (15) and (19) yield

$$C^6 = 0;\quad C^1 = 0;\quad C^2 = C^4,$$
$$\delta_0 = 0;\quad C^2x + C^3 - \xi^0_0 = 0;\quad \delta_{11} = 0;\quad (C^2 - 4C^4)x + C^3 - 2C^5 = 0.$$

Thus,

$$\delta = C^8x + C^9;\quad -3C^4 = 0;\quad C^3 = 2C^5;\quad C^2 = 0,$$

and we derive the coefficients of the vector fields (3) in the following form

$$\xi^0 = C^3t + C^{10} = 2C^5t + C^{10};\quad \xi^1 = C^5x + C^7;\quad \eta = 2C^5u + C^8x + C^9.$$

Hence, Eq. (2) with an arbitrary right-hand side H admits 5-dimensional Lie algebra $\mathfrak{g}^0$, spanned by the vector fields

$$Q^1 = \partial_t;\quad Q^2 = \partial_x;\quad Q^3 = \partial_u;\quad Q^4 = 2t\partial_t + x\partial_x + 2u\partial_u;\quad Q^5 = x\partial_u.$$

Therefore, the following statement holds.

Theorem 2 *The kernel Lie invariance algebra of equations from the class (2) is* $\mathfrak{g}^0$.

2 Group Classification

According to the classical Lie theorem on Lie algebras of vector fields on the real line [8, Satz 6, Seite 455] (see also [13, Theorem 2.70] and [4, Theorem 1]), nonequivalent realizations of finite-dimensional Lie algebras by vector fields on the t-line are exhausted by the algebras

$$\{0\},\quad \langle\partial_t\rangle,\quad \langle\partial_t,\, t\partial_t\rangle,\quad \langle\partial_t,\, t\partial_t,\, t^2\partial_t\rangle.$$

Denote by π the projection from $\mathbb{R}^t \times \mathbb{R}^x$ onto $\mathbb{R}^t$, and let

$$k := \dim \pi_*\mathfrak{g}^H,$$

where $\mathfrak{g}^H$ is the Lie invariance algebra of the equation from the class (2).

It is obvious that $\dim \pi_*\mathfrak{g}^0 = 2$ for the kernel Lie invariance algebra $\mathfrak{g}^0$, therefore, for any equation from the class (2), either $k = 2$ or $k = 3$, that is,

$$\pi_*\mathfrak{g}^H = \langle\partial_t,\, t\partial_t\rangle \quad\text{or}\quad \pi_*\mathfrak{g}^H = \langle\partial_t,\, t\partial_t,\, t^2\partial_t\rangle.$$

Below, we consider each of these two cases separately.

Firstly, differentiating (19) twice with respect to x, we derive

$$\delta_{011} - \delta_{1111}H' = 0,$$

which yields $\delta_{011} = 0$, $\delta_{1111} = 0$, and, therefore,

$$\delta = C^{20}x^3 + C^{21}x^2 + \rho^1(t)x + \rho^0(t), \tag{20}$$

where $\rho^1(t)$ and $\rho^0(t)$ run through the set of smooth functions of variable t.

$\boldsymbol{k = 3}$. In this case

$$\xi^0 = \lambda^2t^2 + \lambda^1t + \lambda^0, \tag{21}$$

where λ^2, λ^1, and λ^0 are arbitrary constants. Substituting (20) and (21) into (19), we get

$$\rho^1_0x + \rho^0_0 + (C^2x + C^3 - 2\lambda^2t - \lambda^1)H - (6C^{20}x + 2C^{21} + (C^2x + C^3 - 4C^4x - 2C^5)u_{11})H' = 0.$$

Splitting this equation with respect to x and 1 yields the following system

$$\rho^1_0 + C^2H - (6C^{20} + (C^2 - 4C^4)u_{11})H' = 0,$$
$$\rho^0_0 + (C^3 - 2\lambda^2t - \lambda^1)H - (2C^{21} + (C^3 - 2C^5)u_{11})H' = 0.$$

Differentiating the second equation of the above system with respect to t, we obtain

$$\rho^0_{00} - 2\lambda^2H = 0,$$

hence $\lambda^2 = 0$, since H is a nonlinear function of u_{11}. Therefore, the function ξ^0 can be at most linear. This contradicts the condition $k = 3$.

$\boldsymbol{k = 2}$. In this case ξ^0 is linear, that is,

$$\xi^0 = \lambda^1t + \lambda^0, \tag{22}$$

where λ^1 and λ^0 are arbitrary constants. Substituting (20) and (22) into (19), we have

$$\rho^1_0x + \rho^0_0 + (C^2x + C^3 - \lambda^1)H - (6C^{20}x + 2C^{21} + (C^2x + C^3 - 4C^4x - 2C^5)u_{11})H' = 0.$$

Splitting this equation with respect to x and 1, we derive the following system

$$\rho_0^1 + C^2 H - (6C^{20} + (C^2 - 4C^4)u_{11})H' = 0, \tag{23}$$
$$\rho_0^0 + (C^3 - \lambda^1)H - (2C^{21} + (C^3 - 2C^5)u_{11})H' = 0. \tag{24}$$

Differentiating (23) and (24) with respect to t gives $\rho_{00}^1 = 0$ and $\rho_{00}^0 = 0$, respectively. Therefore, $\rho^1(t) = C^{22}t + C^{23}$ and $\rho^0(t) = C^{24}t + C^{25}$. Hence,

$$\delta(t,x) = C^{20}x^3 + C^{21}x^2 + (C^{22}t + C^{23})x + (C^{24}t + C^{25}).$$

From (16)–(18), we have the following expressions for components of Q

$$\xi^0 = \lambda^1 t + \lambda^0; \quad \xi^1 = C^1 u + C^4 x^2 + C^5 x + C^6 t + C^7,$$
$$\eta = (C^2 x + C^3)u + C^{20}x^3 + C^{21}x^2 + (C^{22}t + C^{23})x + (C^{24}t + C^{25}),$$

while from (15), (23) and (24) , we derive the system of the classifying equations for function H in the form

$$C^6 + C^1 H + (2C^2 - 2C^4 - 3C^1 u_{11})H' = 0, \tag{25}$$
$$C^{22} + C^2 H - (6C^{20} + (C^2 - 4C^4)u_{11})H' = 0, \tag{26}$$
$$C^{24} + (C^3 - \lambda^1)H - (2C^{21} + (C^3 - 2C^5)u_{11})H' = 0. \tag{27}$$

All these equations have the form

$$a + bH + cH' + du_{11}H' = 0, \tag{28}$$

where a, b, c and d are some constant parameters. Up to the equivalence defined in Theorem 1, there exist only three possibilities for the solutions of Eq. (28), namely, $e^{u_{11}}$, $\ln(u_{11})$, and u_{11}^p with $p \neq 0, 1$.

If $H = e^{u_{11}}$, then after substitution into system (25)–(27), we derive

$$C^6 + C^1 e^{u_{11}} + 2(C^2 - C^4)e^{u_{11}} - 3C^1 u_{11}e^{u_{11}} = 0,$$
$$C^{22} + C^2 e^{u_{11}} - 6C^{20}e^{u_{11}} - (C^2 - 4C^4)u_{11}e^{u_{11}} = 0,$$
$$C^{24} + (C^3 - \lambda^1)e^{u_{11}} - 2C^{21}e^{u_{11}} - (C^3 - 2C^5)u_{11}e^{u_{11}} = 0.$$

Splitting this system, we come to the following conditions for the constants

$$C^3 = 2C^5; \quad \lambda^1 = 2C^5 - 2C^{21}; \quad C^1 = C^2 = C^4 = C^6 = C^{20} = C^{22} = C^{24} = 0.$$

Therefore,

$$\xi^0 = 2(C^5 - C^{21})t + \lambda^0; \quad \xi^1 = C^5 x + C^7; \quad \eta = 2C^5 u + C^{21}x^2 + C^{23}x + C^{25}.$$

If $H = \ln(u_{11})$, then system (25)–(27) gives

$$C^6 + C^1\ln(u_{11}) + \frac{2(C^2 - C^4)}{u_{11}} - 3C^1 = 0,$$
$$C^{22} + C^2\ln(u_{11}) - \frac{6C^{20}}{u_{11}} - (C^2 - 4C^4) = 0,$$
$$C^{24} + (C^3 - \lambda^1)\ln(u_{11}) - \frac{2C^{21}}{u_{11}} - (C^3 - 2C^5) = 0.$$

Splitting of this system leads us to the following restrictions for the constants

$$\lambda^1 = C^3; \quad C^{24} = C^3 - 2C^5; \quad C^1 = C^2 = C^4 = C^6 = C^{20} = C^{21} = C^{22} = 0.$$

Hence,

$$\xi^0 = C^3 t + \lambda^0; \quad \xi^1 = C^5 x + C^7; \quad \eta = C^3 u + C^{23}x + (C^3 - 2C^5)t + C^{25}.$$

If $H = u_{11}^p$, then system (25)–(27) yields

$$C^6 + C^1 u_{11}^p + 2(C^2 - C^4)pu_{11}^{p-1} - 3C^1 pu_{11}^p = 0,$$
$$C^{22} + C^2 u_{11}^p - 6C^{20}pu_{11}^{p-1} - (C^2 - 4C^4)pu_{11}^p = 0,$$
$$C^{24} + (C^3 - \lambda^1)u_{11}^p - 2C^{21}pu_{11}^{p-1} - (C^3 - 2C^5)pu_{11}^p = 0.$$

Splitting this system, we obtain

$$C^1 = 3pC^1; \quad C^2 = C^4; \quad C^2 = p(C^2 - 4C^4); \quad \lambda^1 = (1-p)C^3 + 2pC^5;$$
$$C^6 = C^{20} = C^{21} = C^{22} = C^{24} = 0.$$

If $p \neq \pm\frac{1}{3}$, then

$$\lambda^1 = (1-p)C^3 + 2pC^5; \quad C^1 = C^2 = C^6 = C^{20} = C^{21} = C^{22} = C^{24} = 0$$

and, therefore, components of the vector field Q look like

$$\xi^0 = ((1-p)C^3 + 2pC^5)t + \lambda^0; \quad \xi^1 = C^5 x + C^7; \quad \eta = C^3 u + C^{23}x + C^{25}.$$

For $p = \frac{1}{3}$,

$$\lambda^1 = \frac{2}{3}(C^3 + C^5); \quad C^2 = C^6 = C^{20} = C^{21} = C^{22} = C^{24} = 0,$$

while components of the vector field Q are of the following form

$$\xi^0 = \frac{2}{3}(C^3 + C^5)t + \lambda^0; \quad \xi^1 = C^1 u + C^5 x + C^7; \quad \eta = C^3 u + C^{23}x + C^{25}.$$

Table 1 The result of group classification of the class (2)

$H(u_{xx})$	Lie invariance algebra
$\forall$	$\langle\partial_t,\ \partial_x,\ \partial_u,\ 2t\partial_t+x\partial_x+2u\partial_u,\ x\partial_u\rangle$
$\exp u_{xx}$	$\langle\partial_t,\ \partial_x,\ \partial_u,\ 2t\partial_t+x\partial_x+2u\partial_u,\ x\partial_u,\ 2t\partial_t-x^2\partial_u\rangle$
$\ln u_{xx}$	$\langle\partial_t,\ \partial_x,\ \partial_u,\ 2t\partial_t+x\partial_x+2u\partial_u,\ x\partial_u,\ x\partial_x-2t\partial_u\rangle$
$u_{xx}^p,\ p\neq 0,\pm\frac{1}{3},1$	$\langle\partial_t,\ \partial_x,\ \partial_u,\ 2t\partial_t+x\partial_x+2u\partial_u,\ x\partial_u,\ (1-p)t\partial_t+u\partial_u\rangle$
$u_{xx}^{1/3}$	$\langle\partial_t,\ \partial_x,\ \partial_u,\ 2t\partial_t+x\partial_x+2u\partial_u,\ x\partial_u,\ 2t\partial_t+3u\partial_u,\ u\partial_x\rangle$
$u_{xx}^{-1/3}$	$\langle\partial_t,\ \partial_x,\ \partial_u,\ 2t\partial_t+x\partial_x+2u\partial_u,\ x\partial_u,$ $4t\partial_t+3u\partial_u,\ x^2\partial_x+xu\partial_u\rangle$

For $p=-\frac{1}{3}$,

$$\lambda^1=\frac{2}{3}(2C^3-C^5);\quad C^1=C^6=C^{20}=C^{21}=C^{22}=C^{24}=0,$$

and then components of the vector field Q are

$$\xi^0=\frac{2}{3}(2C^3-C^5)t+\lambda^0;\quad \xi^1=C^2x^2+C^5x+C^7,$$
$$\eta=(C^2x+C^3)u+C^{23}x+C^{25}.$$

We summarize the obtained results in the following theorem.

Theorem 3 (The result of group classification, [1, Sect. 4]) *A complete list of $G^{\sim}$-inequivalent (maximal) Lie-symmetry extensions in the class (2) is exhausted by the following cases:*

(0) *general case* $H=H(u_{11})$, $\quad\mathfrak{g}^0=\langle Q^1,Q^2,Q^3,Q^4,Q^5\rangle$,

(1) $H(u_{11})=\exp(u_{11})$, $\quad\mathfrak{g}^{\exp(u_{11})}=\langle Q^1,Q^2,Q^3,Q^4,Q^5,Q^{6a}\rangle$,

(2) $H(u_{11})=\ln(u_{11})$, $\quad\mathfrak{g}^{\ln(u_{11})}=\langle Q^1,Q^2,Q^3,Q^4,Q^5,Q^{6b}\rangle$,

(3) $H(u_{11})=u_{11}^p,\quad p\neq 0,\pm\dfrac{1}{3},1$, $\quad\mathfrak{g}^{u_{11}^p}=\langle Q^1,Q^2,Q^3,Q^4,Q^5,Q^{6c}\rangle$,

(4) $H(u_{11})=u_{11}^{1/3}$, $\quad\mathfrak{g}^{u_{11}^{1/3}}=\langle Q^1,Q^2,Q^3,Q^4,Q^5,Q^{6c_1},Q^{7c_1}\rangle$,

(5) $H(u_{11})=u_{11}^{-1/3}$, $\quad\mathfrak{g}^{u_{11}^{-1/3}}=\langle Q^1,Q^2,Q^3,Q^4,Q^5,Q^{6c_2},Q^{7c_2}\rangle$.

In Table 1, we present the results of the group classification of the class (2).

3 Conclusion and Discussion

In the present paper, we have reconsidered the complete group classification of the class of the (1+1)-dimensional nonlinear evolution equations (2) up to the $G^{\sim}$-equivalence that firstly had been performed by Akhatov, Gazizov and Ibragimov in the famous paper [1, Sect. 4]. In our modified classification procedure, we have used the specific structure of Lie symmetries of evolution equations for involving the classical Lie theorem on realizations of Lie algebras by vector fields on the line. Previously, the Lie theorem has already been applied to the group classification of different classes of both ordinary and partial differential equations (see [2, 4–6, 10, 11, 14] and references therein). This approach has substantially simplified the proof of the classification results and, in particular, we have made the solution of the classifying equations easier.

It is important to note that, according to [12, Theorem 0.1], the upper bound of the dimensions of Lie invariance algebras of nonlinearisable (1+1)-dimensional second-order evolution equations is equal to 7, and moreover, according to [12, Theorem 3.5], all such equations with the 7-dimensional Lie invariance algebras are equivalent (up to contact transformations) to the equation $u_t=u_{xx}^{-1/3}$ (case (5) of Theorem 3). Gazizov [7] and Pukhnachov [15] found the explicit contact transformations that map case (4) to case (5) (see the corresponding discussion in [3]).

In the future work, we plan to study the contact- and point-transformation structures associated with class (2), as well as to perform the classification of subalgebras of the algebras from Theorem 3 and the corresponding Lie reductions, etc.

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