ARTICLE

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UNEXPECTED EFFECT OF EULER'S FORMULAS

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Summary. *This paper explores significant aspects of geometry, specifically problems and theorems related to two classical Euler's formulas: $IO^2 = R^2 - 2Rr$, which describes the distance between the centers of the circumcircle and the incircle, and $(OI_a)^2 = R^2 + 2Rr_a$, which characterizes the distance between the centers of the circumcircle and the A-excircle. The author notes that, despite their importance, one of the problems proposed by S. I. Zettel in his book Problems on Maxima and Minima has been largely overlooked in the mathematical community. The paper demonstrates how the application of formulas for the radii of the incircle and excircle, $r_a = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$ and $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$, not only simplifies the solution of the problem but also leads to a new extension of this result. The key idea is the use of the "analogy" method, which allows the discovery of new relationships and makes this approach appealing and useful for a broad range of mathematical researchers. Additionally, the paper includes a discussion of theorems and lemmas that will be applied in the proofs of the results, with the expectation that the material will be practically useful for readers.*

Keywords: *Euler's formulas, circumcircle, excircle, incircle, Mansions circle, Trillium theorem, law of cosines, law of sines, analogy.*

Let us consider the triangle ΔAIK_3.
The length of AI is given by: $AI = \frac{IK_3}{\sin\frac{\angle BAC}{2}} = \frac{r}{\sin\frac{\angle A}{2}}$,
where r is the inradius of the triangle ABC (Fig. 1).
According to the *Trillium Theorem* [1], we have:
$IW_1 = CW_1 = BW_1$.

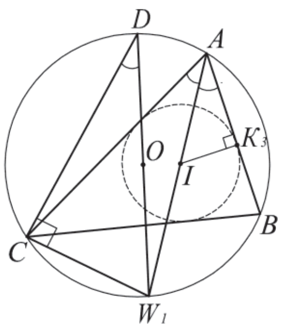

Fig. 1

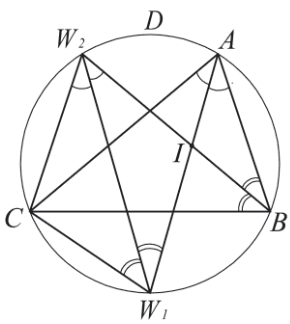

Fig. 2

Also, we have $IW_1 = CW_1$ (Fig. 2).
For triangle ΔCW_2W_1, we have:
$\Delta CW_2W_1 = \Delta IW_2W_1$, since they share the common side W_2W_1 and two adjacent equal angles.
Moreover, $CW_1 = BW_1$, as the chords subtend equal arcs.

In triangle ΔCDW_1, we have: $CW_1 = 2Rsin\frac{\angle BAC}{2} = 2Rsin\frac{\angle A}{2}$,
where R is the circumradius of triangle ABC.

Therefore, we can express:

$AI \cdot IW_1 = AI \cdot CW_1 = \frac{r}{sin\frac{\angle A}{2}} \cdot 2Rsin\frac{\angle A}{2}$ which simplifies to [4]:

$$\boxed{AI \cdot IW_1 = 2Rr} \quad (1)$$

Let $\gamma = (O; R = OA)$ be the circumcircle of triangle ABC.

Let $\gamma_M = (W_1;\ R_M = IW_1)$ be the Mansions circle.

Let $\gamma_a = (I_a;\ r_a = I_aT_2)$ be the excircle, tangent to side BC and the extensions of sides AC and AB (Fig. 3).

We also know that:

$$W_1 = CW_1 = BW_1 = W_1I_a = R_M,$$

where R_M is the radius of the Mansions circle, which is the circumcircle of triangle BIC, with I being the incenter, the point of intersection of the angle bisectors of triangle ABC.

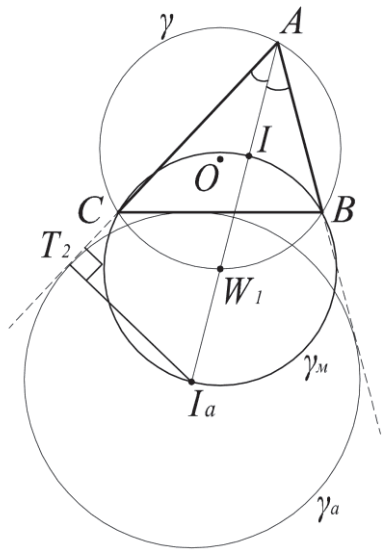

Fig. 3

Next, we have:

$AI_a = \frac{I_aT_2}{\sin\frac{\angle BAC}{2}} = \frac{r_a}{\sin\frac{\angle A}{2}}$ (ΔAT_2I_a) (see Fig. 3).

Thus, we can write:

$$AI_a \cdot W_1I_a = \frac{r_a}{\sin\frac{\angle A}{2}} \cdot 2Rsin\frac{\angle A}{2} = 2Rr_a \text{ [3]},$$

which simplifies to:

$$\boxed{AI_a \cdot W_1I_a = 2Rr_a} \quad (2)$$

Below is the proof of the first Euler's formula (the distance between the centers of the circumscribed and the inscribed circles around a triangle):

$$(OI)^2 = R^2 - 2Rr$$

(where R is the radius of the circumcircle around triangle ABC, and r is the radius of the incircle of triangle ABC) (Fig. 4).

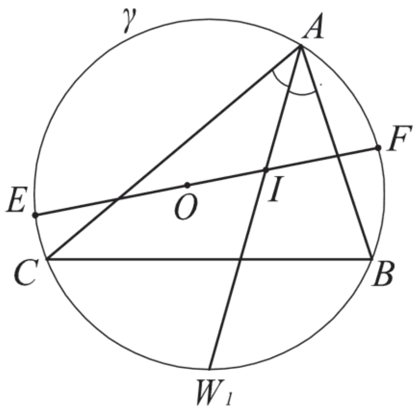

Fig. 4

Proof:

Using the chord segment product theorem (see Fig. 4), we have: $EI \cdot IF = AI \cdot IW_1$

Where $EI = R + OI$ and $IF = R - OI$.

Thus, $(R + OI)(R - OI) = AI \cdot IW_1$

Simplifying, $R^2 - (OI)^2 = AI \cdot IW_1$

Using equation (1), we get: $R^2 - (OI)^2 = 2Rr$

or $R^2 - 2Rr = (OI)^2$

This is the first Euler's formula.

Q.E.D.

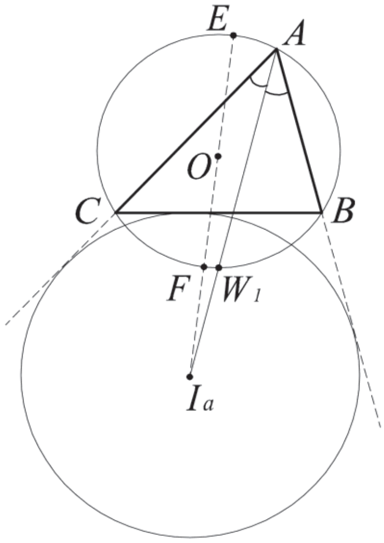

Fig. 5

Now, we will prove the second Euler's formula (the distance between the centers of the circumscribed and the exscribed circles of the triangle):

$$(OI_a)^2 = R^2 + 2Rr_a$$

(where r_a is the radius of the excircle opposite vertex A,
tangent to side BC and the extensions of sides AC and AB of triangle ABC) (Fig. 5).

Proof:

By the tangent-secant theorem, which states that the square of the tangent equals the product of the secant segment and its external part, we have:

$$\boxed{I_aA \cdot I_aW_1 = I_aE \cdot I_aF} \quad (3)$$

where I_aA and I_aE are secants to the circle $\gamma = (O; R = OA)$.

Using formula (2), we get:

$$I_aE = OI_a + OE = OI_a + R \text{ and } I_aF = OI_a - OF = OI_a - R$$

Thus, the product $I_aE \cdot I_aF$ becomes:

$$I_aE \cdot I_aF = (OI_a + R)(OI_a - R)$$
$$I_aE \cdot I_aF = (OI_a)^2 - R^2$$

Rewriting equation (3), we get: $2Rr_a = (OI_a)^2 - R^2$

or $R^2 + 2Rr_a = (OI_a)^2$

This is the second Euler's formula.

Q.E.D.

Let us now consider a few interesting dependencies:

$$AK_2 = AK_3 = p - a$$
$$BK_1 = BK_3 = p - b$$
$$CK_1 = CK_2 = p - c$$

(where p is the semiperimeter of triangle ABC, and a, b, c are the sides BC, AC, and AB of triangle ABC, respectively) (Fig. 6).

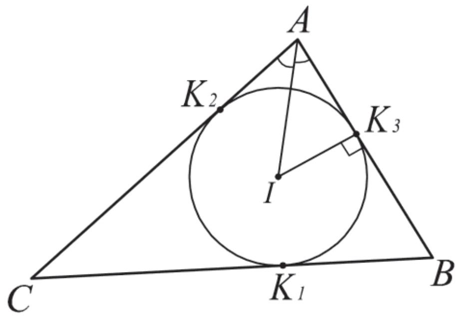

Fig. 6

Now, consider the triangle ΔAIK_3.

We have: $IK_3 = r = AK_3 \cdot tg\frac{\angle BAC}{2} = (p - a)tg\frac{\angle A}{2}$

This can be rewritten as: $r = \frac{b+c-a}{2} \cdot tg\frac{\angle A}{2} = \frac{2R(sin\angle B + sin\angle C - sin\angle A)}{2} \cdot tg\frac{\angle A}{2} =$

$= R(sin\angle B + sin\angle C - sin\angle A) \cdot \frac{sin\frac{\angle A}{2}}{cos\frac{\angle A}{2}} = 4R \cdot sin\frac{\angle A}{2} \cdot sin\frac{\angle B}{2} \cdot sin\frac{\angle C}{2}.$

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Thus, we obtain the formula:

$$\boxed{r = 4R\sin\frac{\angle A}{2}\sin\frac{\angle B}{2}\sin\frac{\angle C}{2}} \quad (4)$$

Next, we consider the following dependency:

$AC = b, AB = c$

$CT_2 = CT_1 = p - b$

$BT_1 = BT_3 = p - c$

$AT_2 = AT_3 = p$

(where $p = \frac{a+b+c}{2}$) (Fig. 7).

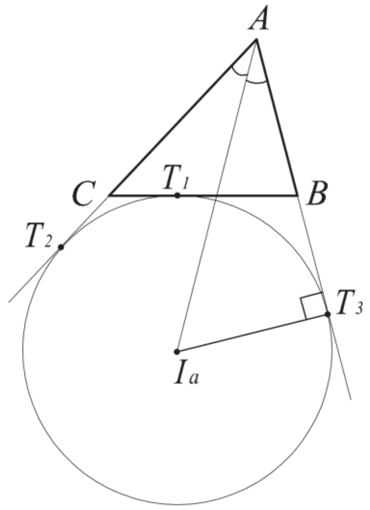

Fig. 7

From the triangle ΔAI_aT_3, we get:
$I_aT_3 = r_a = AT_3 tg\frac{\angle BAC}{2} = p \cdot tg\frac{\angle A}{2}$.
The formula for $r_a = p \cdot tg\frac{\angle A}{2}$ can be rewritten in a form analogous to equation (4):

$$r_a = p \cdot tg\frac{\angle A}{2} = \frac{a+b+c}{2} \cdot \frac{\sin\frac{\angle A}{2}}{\cos\frac{\angle A}{2}} = \frac{2R(\sin\angle A + \sin\angle B + \sin\angle C)}{2} \cdot \frac{\sin\frac{\angle A}{2}}{\cos\frac{\angle A}{2}} =$$

$$= 4R\cos\frac{\angle A}{2}\cos\frac{\angle B}{2}\cos\frac{\angle C}{2} \cdot \frac{\sin\frac{\angle A}{2}}{\cos\frac{\angle A}{2}} = 4R\sin\frac{\angle A}{2}\cos\frac{\angle B}{2}\cos\frac{\angle C}{2}.$$

Thus, we obtain the formula:

$$\boxed{r_a = 4R\sin\frac{\angle A}{2}\cos\frac{\angle B}{2}\cos\frac{\angle C}{2}} \quad (5)$$

In 1948, S.I. Zetel [2] proposed the following problem:

In triangle ABC, with the smallest side $BC = a$, segments CE and BF are drawn from vertices C and B along sides CA and BA respectively, such that $BC = a$. Prove that the radius of the circumcircle of triangle AEF is equal to the distance between the incenter (the point of intersection of the angle bisectors of triangle ABC) and the circumcenter (the center of the circumcircle of triangle ABC).

$\gamma = (O; OA = R)$

$\vartheta = (I; r = IK_3)$ (Fig. 8)

$\gamma_o = (Q;\ R_o = QA)$ (Fig. 9)

Alternatively, we need to prove that: $R_o = \sqrt{R^2 - 2Rr}$.

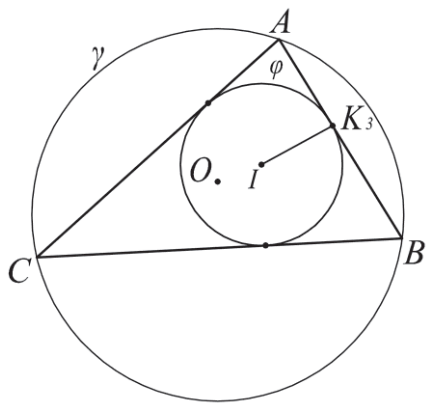

Fig. 8

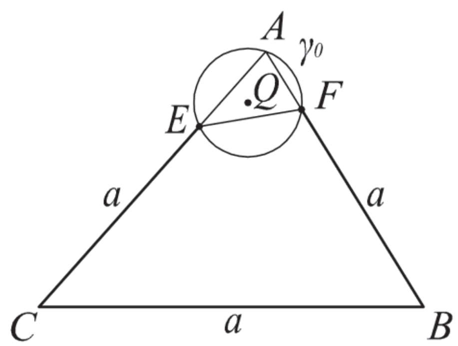

Fig. 9

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Proof:

From the isosceles triangle CBF (Fig. 10), we get:

$CF = 2a \cdot sin\frac{\angle CBA}{2} = 2a \cdot sin\frac{\angle B}{2}$.

From triangle CEF, using the cosine rule:

$(EF)^2 = 4a^2 sin^2\frac{\angle B}{2} + a^2 - 4a^2 \cdot sin\frac{\angle B}{2} \cdot cos\angle FCE$.

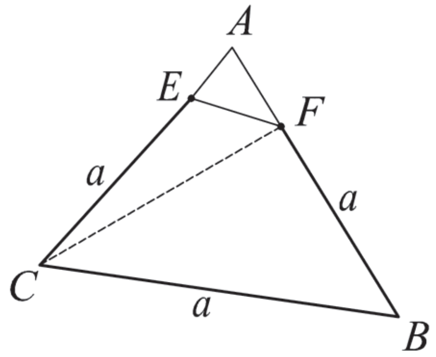

Fig. 10

Since the angle $\angle FCE$ is the difference between angles $\angle BCA$ and $\angle BCF$, we have:

$\angle FCE = \angle C - \left(90° - \frac{\angle B}{2}\right) = \frac{|\angle C - \angle A|}{2}$.

Therefore: $(EF)^2 = 4a^2 sin^2\frac{\angle B}{2} + a^2 - 4a^2 sin\frac{\angle B}{2} \cdot cos\frac{\angle C-\angle A}{2}$.

Thus, $(EF)^2 = a^2\left(4sin^2\frac{\angle B}{2} + 1 - 4sin\frac{\angle B}{2} cos\frac{\angle C-\angle A}{2}\right) =$

$= a^2(1 + 4sin\frac{\angle B}{2}(sin\frac{\angle B}{2} - cos\frac{\angle C-\angle A}{2}))$;

or equivalently: $(EF)^2 = a^2(1 - 8sin\frac{\angle A}{2} sin\frac{\angle B}{2} sin\frac{\angle C}{2})$.

By using formula (4), we obtain: $(EF)^2 = a^2\left(1 - \frac{2r}{R}\right) = a^2\left(\frac{R-2r}{R}\right)$

By the law of sines in triangle ABC: $a^2 = 4R^2 sin^2\angle BAC = 4R^2 sin^2\angle A$

Thus, $(EF)^2 = 4R^2 sin^2\angle A\left(\frac{R-2r}{R}\right) = 4sin^2\angle A(R^2 - 2Rr)$

$$\boxed{(EF)^2 = 4sin^2\angle A(R^2 - 2Rr)} \quad (6)$$

On the other hand, from triangle AEF, using the law of sines, we get:

$(EF)^2 = 4R_o^2 sin^2\angle BAC = 4R_o^2 sin^2\angle A$

$$\boxed{(EF)^2 = 4R_o^2 sin^2\angle A} \quad (7)$$

Comparing equations (6) and (7), we obtain:

$4sin^2\angle A(R^2 - 2Rr) = 4R_o^2 sin^2\angle A$

Simplifying, $R^2 - 2Rr = R_o^2$

Thus, $\sqrt{R^2 - 2Rr} = R_o$

The first Euler's formula, which was proven earlier, states that:

$(OI)^2 = R^2 - 2Rr$ or $OI = \sqrt{R^2 - 2Rr}$.

Hence, it has been proven that

$R_o = OI = \sqrt{R^2 - 2Rr}$.

Q.E.D.

Below is a formulation and proof of a similar problem that has not been encountered in the professional literature before:

In triangle ABC, with the smallest side $BC = a$, segments CE and BF are drawn along the extensions of sides AC and AB from vertices C and B such that

$CE = BF = a$. Prove that the radius of the circumcircle of triangle AEF is equal to the distance between the circumcenter of triangle ABC and the center of the excircle that is tangent to side BC and the extensions of sides AC and AB.

$\gamma = (O; R = OA)$

$\gamma_a = (I_a;\ r_a = I_aT_3)$ (Fig. 11)

$\gamma_o = (Q;\ R_Q = QE)$ (Fig. 12)

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Alternatively, we need to prove that: $R_Q = OI_a = \sqrt{R^2 + 2Rr_a}$.

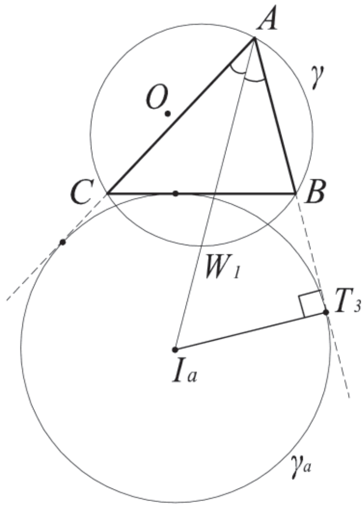

Fig. 11

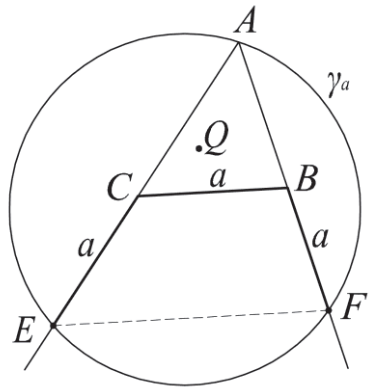

Fig. 12

Proof:

From the isosceles triangle CBF (Fig. 13), we have:

$CF = 2a \cdot cos\frac{\angle ABC}{2} = 2a \cdot cos\frac{\angle B}{2}$.

From triangle CEF, using the cosine rule:

$(EF)^2 = 4a^2 \cdot cos^2\frac{\angle B}{2} + a^2 - 4a^2 \cdot cos\frac{\angle B}{2} \cdot cos\angle ECF$.

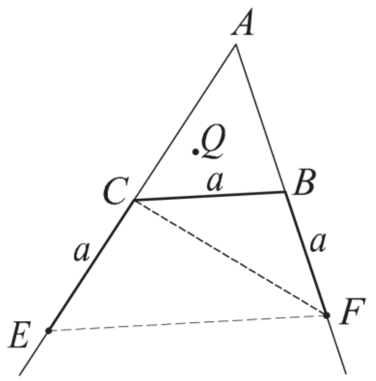

Fig. 13

Since the angle $\angle ECF$ is the difference between angles $\angle BCE$ and $\angle BCF$, we get:

$\angle ECF = 180° - \angle C - \frac{\angle B}{2} = 90° - \frac{\angle C - \angle A}{2}$.

Thus, $(EF)^2 = 4a^2 \cdot cos^2\frac{\angle B}{2} + a^2 - 4a^2 \cdot cos\frac{\angle B}{2} \cdot sin\frac{\angle C - \angle A}{2}$.

Therefore,

$(EF)^2 = a^2(1 + 4cos\frac{\angle B}{2}\left(cos\frac{\angle B}{2} - sin\frac{\angle C - \angle A}{2}\right) = a^2\left(1 + 4cos\frac{\angle B}{2}\left(sin\frac{\angle C + \angle A}{2} - \right.\right.$

$\left.\left.-sin\frac{\angle C - \angle A}{2}\right)\right) = a^2(1 + 4sin\frac{\angle B}{2}cos\frac{\angle B}{2}cos\frac{\angle C}{2})$.

Using formula (5), we get:

$(EF)^2 = a^2\left(1 + \frac{2r_a}{R}\right) = a^2\left(\frac{R + 2r_a}{R}\right)$,

Since $a^2 = 4R^2sin^2\angle A$ by the law of sines for triangle ABC, we obtain:

$$\boxed{(EF)^2 = 4sin^2\angle A(R^2 + 2Rr_a)} \qquad (8)$$

From triangle AEF, using the law of sines, we get:

$$\boxed{(EF)^2 = 4R_Q^2 sin^2\angle A} \qquad (9)$$

Comparing equations (8) and (9), we get: $4sin^2\angle A(R^2 + 2Rr_a) = 4R_Q^2 sin^2\angle A$

Simplifying: $R^2 + 2Rr_a = R_Q^2$

Thus, $\sqrt{R^2 + 2Rr_a} = R_Q$

The second Euler's formula, which was proven earlier, is:

$(OI_a)^2 = R^2 + 2Rr_a$ або $OI_a = \sqrt{R^2 + 2Rr_a}$.

Therefore, it has been proven that
$R_Q = OI_a = \sqrt{R^2 + 2Rr_a}$.
Q.E.D.

There are other ways to prove the proposed problems. I suggest exploring them independently.

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НЕСПОДІВАНИЙ ЕФЕКТ ФОРМУЛ ЕЙЛЕРА

Гетманенко Людмила Миколаївна
старша викладачка кафедри природничо-математичної освіти і технологій
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Анотація. *У статті досліджуються важливі аспекти геометрії, зокрема задачі та теореми, що стосуються двох класичних формул Ейлера: $IO^2 = R^2 - 2Rr$, яка описує відстань між центрами описаного та вписаного кіл, та $(OI_a)^2 = R^2 + 2Rr_a$, яка характеризує відстань між центрами описаного та зовнівписаного кіл. Автор відзначає, що попри їх значущість, одна з задач С. І. Зетеля, представлена у книзі «Задачі на максимум і мінімум», залишалася непоміченою в математичній спільноті. В статті показано, як застосування формул для радіусів вписаного та зовнівписаного кіл, $r_a = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$ та $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$, дозволяє не лише спростити розв'язок задачі, але й знайти нове продовження цього результату. Основною ідеєю є застосування методу «аналогії», який дозволяє знаходити нові залежності та робить цей підхід привабливим і корисним для широкого кола математичних дослідників. Окрім цього, стаття містить розгляд теорем і лем, які будуть використовуватися для доказу отриманих результатів, та сподівається на практичне застосування матеріалу читачами.*

Ключові слова: *формули Ейлера, описане коло, зовнівписане коло, вписане коло, коло Мансіона, теорема Трилисника, теорема косинусів, теорема синусів, аналогія.*