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ARTICLE

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PROPERTIES OF THE BITANGENT CIRCLE

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Summary. *This paper explores the properties of the bitangent circle, which is defined as a circle that is tangent to two sides of a triangle and to the circumcircle of the triangle. The problem related to such a circle was first proposed at the 20th International Mathematical Olympiad (1976, Bucharest) for an isosceles triangle. However, it was later discovered that the problem's condition is general for any triangle. The study of this problem attracted the attention of mathematicians from various countries, and numerous attempts at a proof emerged. This paper presents a new approach to solving the problem using formulas that simplify the proof compared to other methods, such as Archimedes' lemma. Additionally, the paper examines specific properties of the bitangent circle, which can form the basis for problems of increased difficulty. The content of the paper will interest mathematicians, educators, students, and all those passionate about geometry.*
Keywords: *bitangent circle; internal tangency of circles; incenter of a triangle; law of cosines; Trillium theorem.*

Problem 1.

A circle with center I_1 is tangent to the sides AC and AB of triangle ABC at points P and Q, respectively, and is also internally tangent to the circumcircle of triangle ABC [4]. Prove that the incenter I of this triangle coincides with the midpoint of segment PQ.

In triangle ABC (Fig. 1), points P and Q are the tangency points of the bidotangent circle (denoted as $\gamma_6 = (I_1; r_1 = I_1P)$), O is the center of the circumcircle of triangle ABC ($\gamma = (O; R = OA)$), and I is the incenter of triangle ABC (the intersection of its angle bisectors) (Fig. 1).

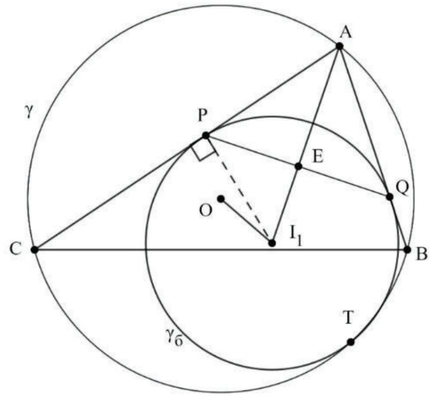

Fig. 1

Preliminary Formulas and Proofs

Consider the incircle $\gamma_0 = (I, r = IK_3)$ of triangle ABC, where K_3 is the tangency point of incircle with center I, the incenter. Examining triangle AIK_3 (Fig. 2):

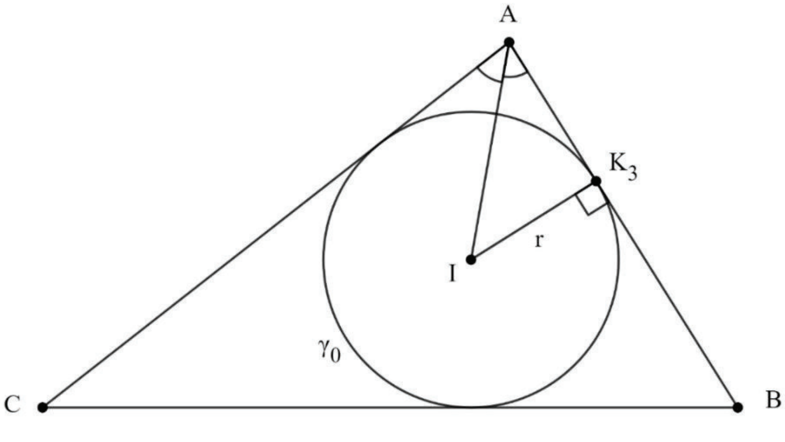

Fig. 2

$$AI = \frac{IK_3}{\sin\frac{\angle BAC}{2}} = \frac{r}{\sin\frac{\angle A}{2}} \qquad (1)$$

The circumcircle $\gamma = (O; R = OA)$ contains the inscribed triangle ABC. The angle bisector AL_1 of angle $\angle BAC$ intersects the circumcircle at point W_1. The altitude AH_1 of triangle ABC is drawn to side BC. The angle H_1AL_1 (between the altitude and the bisector from vertex A) satisfies:

$\frac{|\angle ABC - \angle ACB|}{2} = \frac{|\angle B - \angle C|}{2}$

Assuming$\angle ABC > \angle ACB$ (or equivalently $\angle B > \angle C$), we obtain:

$\angle H_1AL_1 = \frac{\angle B - \angle C}{2} = \varphi$.

Thus, angle is equal to H_1AL_1, since they are alternate interior angles with parallel lines AH_1 and W_1D and transversal AW_1(Fig. 3).

For triangle OAW_1, angles $\angle OW_1A$ and $\angle OAW_1$ are equal, as they are base angles of the isosceles triangle OAW_1 (where $OW_1 = OA$, Fig. 4).

Consequently,

$$\angle OAW_1 = \frac{\angle B - \angle C}{2} = \varphi \qquad (2)$$

Furthermore, in $\triangle DW_1A$ (Fig. 3),

$$AW_1 = 2R \cdot \cos\varphi \qquad (3)$$

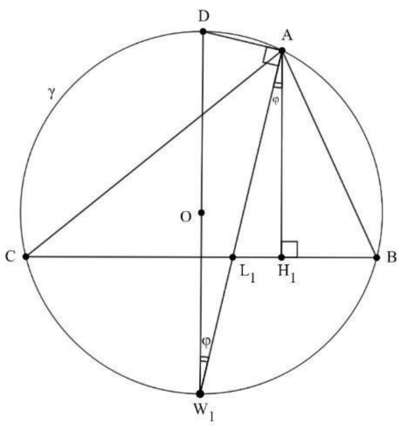

Fig. 3

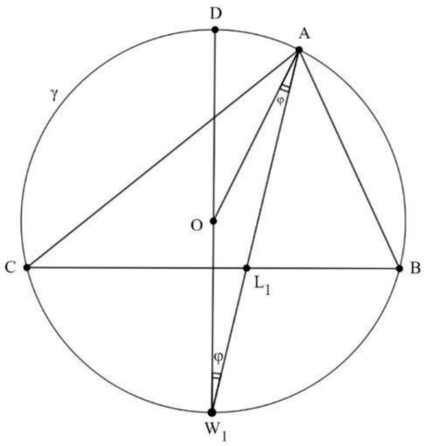

Fig. 4

Chord Relations and the "Trillium Theorem"

From Fig. 5, chords CW_1 and BW_1 are equal because they subtend equal arcs $\overset{\frown}{CW_1}$ and $\overset{\frown}{BW_1}$. By congruence of triangles $\triangle W_2CW_1$ and $\triangle W_2IW_1$ (sharing side W_2W_1 and having two adjacent equal angles), we conclude: $CW_1 = IW_1$

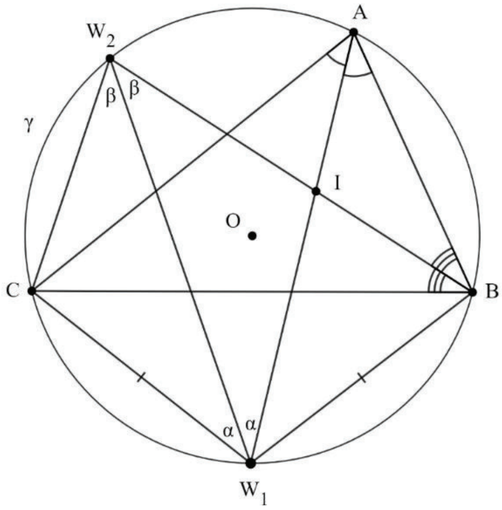

Fig. 5

Thus, the following chord equality holds:

$$CW_1 = BW_1 = IW_1 \tag{4}$$

Isaak Kushnir, in his works, referred to these chords as the "Trillium Theorem" [2].

Additional Proofs

Consider triangle CDW_1. We have:

$$CW_1 = 2R \cdot \sin\frac{\angle A}{2}$$

where $\angle DCW_1 = 90°$, as it subtends a diameter. The angle $\angle CDW_1$ equals $\angle CAW_1$ and also $\angle BAW_1$ (Fig. 6).

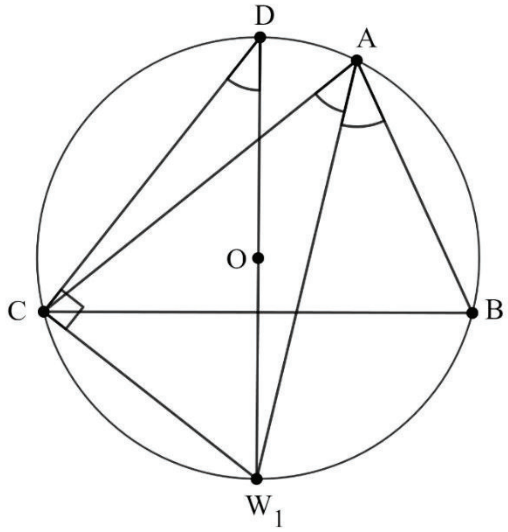

Fig. 6

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To prove Problem 1, we apply the cosine theorem to triangle OAI_1 (Fig. 7). In triangle OAI_1, we have:

- $AO = R$,
- $\angle OAI_1 = \varphi$ (from (2),
- $OI_1 = R - r_1$ (as the distance between the centers of internally tangent circles),
- $AI_1 = \frac{r_1}{\sin\frac{\angle A}{2}}$, since I_1 lies on the bisector AI of angle $\angle BAC$ (Fig. 7).

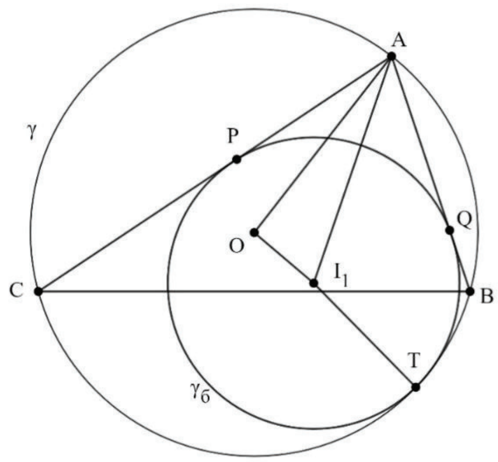

Fig. 7

Applying the cosine theorem to $\triangle OAI_1$:

$$(OI_1)^2 = OA^2 + (AI_1)^2 - 2 \cdot OA \cdot AI_1 \cdot \cos\angle OAI_1$$

$$(R - r_1)^2 = R^2 + \left(\frac{r_1}{\sin\frac{\angle A}{2}}\right)^2 - 2 \cdot R \cdot \frac{r_1}{\sin\frac{\angle A}{2}} \cdot \cos\varphi,$$

Rearranging,

$$R^2 - 2 \cdot R \cdot r_1 + (r_1)^2 = R^2 - 2R \cdot \frac{r_1}{\sin\frac{\angle A}{2}} \cdot cos\varphi + \left(\frac{r_1}{\sin\frac{\angle A}{2}}\right)^2$$

Multiplying through by $sin^2\frac{\angle A}{2}$,

$$(r_1)^2\left(sin^2\frac{\angle A}{2}\right) - 2R \cdot r_1 sin^2\frac{\angle A}{2} = (r_1)^2 - 2R \cdot r_1 \cdot \sin\frac{\angle A}{2} cos\varphi,$$

$$r_1\left(sin^2\frac{\angle A}{2} - 1\right) = \sin\frac{\angle A}{2}\left(2Rsin\frac{\angle A}{2} - 2Rcos\varphi\right),$$

$$r_1 \cdot cos^2\frac{\angle A}{2} = \sin\frac{\angle A}{2}\left(2Rcos\varphi - 2Rsin\frac{\angle A}{2}\right),$$

From (3): $2Rcos\varphi = AW_1$

$2Rsin\frac{\angle A}{2} = CW_1$

From (4): $CW_1 = IW_1$

Thus, $r_1 \cdot cos^2\frac{\angle A}{2} = \sin\frac{\angle A}{2}(AW_1 - IW_1)$ (fig. 5)

$r_1 \cdot cos^2\frac{\angle A}{2} = AI \cdot \sin\frac{\angle A}{2}$

Since, $AI \cdot \sin\frac{\angle A}{2} = r$ (1),

we obtain: $r_1 \cdot cos^2\frac{\angle A}{2} = r$

Solving for r_1, $r_1 = \frac{r}{cos^2\frac{\angle A}{2}}$

which is the radius of the bidotangent circle γ_δ.

Proof of the International Olympiad Problem

The proof of the problem from the International Olympiad (see Fig. 1) is based on the fact that point E coincides with I. In Fig. 1, triangle PAQ is isosceles. The point E lies on AI, where AI_1 is the bisector of angle $\angle BAC$ (since $\triangle PI_1A = \triangle QI_1A$ by the leg $PI_1 = QI_1 = r_1$ and the common hypotenuse).

Thus, AE is both the median and the altitude of triangle PAQ (i.e., $QE = EP$).

We have:

$AE = AP \cdot \cos\frac{\angle A}{2}$ (see Fig. 1, $\triangle AEP$)

$AP = I_1P \cdot ctg\frac{\angle A}{2}$ (Fig. 1, $\triangle API_1$)

$I_1P = r_1$; $AP = r_1 \cdot ctg\frac{\angle A}{2} = \frac{r}{cos^2\frac{\angle A}{2}} \cdot \frac{\cos\frac{\angle A}{2}}{\sin\frac{\angle A}{2}}$

$AP = \frac{r}{\sin\frac{\angle A}{2}\cos\frac{\angle A}{2}}$ (since $\frac{r}{\sin\frac{\angle A}{2}} = AI$ (1))

$AP = \frac{AI}{\cos\frac{\angle A}{2}}$

Thus, $AE = AP \cdot \cos\frac{\angle A}{2} = \frac{AI}{\cos\frac{\angle A}{2}} \cdot \cos\frac{\angle A}{2}$

$AE = AI \Rightarrow E \equiv I$

We have proven that the incenter I of triangle ABC is the midpoint of segment PQ.

Problem 2.

Z.A. Skopets, a renowned world-class geometer, studied the bidotangent circle and proved that: $AP = \frac{bc}{p}$ [1],

where b and c are the sides of $\triangle ABC$ opposite to AC and AB, respectively, and p is the semiperimeter of triangle ABC. He used four circles and Casey's theorem to prove this result.

Given the formula: $AP = \frac{AI}{\cos\frac{\angle A}{2}}$

try to derive this result on your own.

Problem 3 *(Analogue of Euler's Formula).*

Prove that:

$II_1 = AI \cdot tg^2\frac{\angle A}{2}$ [3].

Proof:

$II_1 = AI_1 - AI$ (Fig. 8)

$AI_1 = \frac{AP}{\cos\frac{\angle A}{2}}$ ($\triangle API_1$)

$II_1 = \frac{AP}{\cos\frac{\angle A}{2}} - AI = \frac{AI}{cos^2\frac{\angle A}{2}} - AI$ (since $AP = \frac{AI}{\cos\frac{\angle A}{2}}$)

$II_1 = \frac{AI}{\cos\frac{\angle A}{2}} = \frac{AI\left(1-cos^2\frac{\angle A}{2}\right)}{cos^2\frac{\angle A}{2}} = \frac{AI \cdot sin^2\frac{\angle A}{2}}{cos^2\frac{\angle A}{2}} = AI \cdot tg^2\frac{\angle A}{2}$.

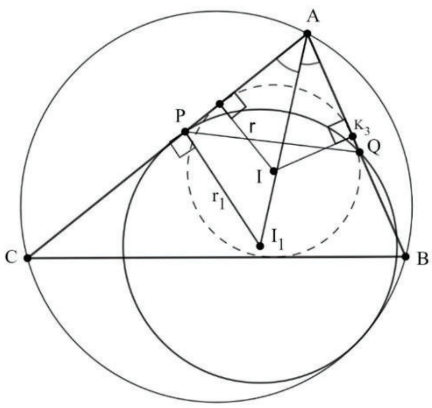

Fig. 8

Thus, the proof is complete

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ВЛАСТИВОСТІ БІДОТИЧНОГО КОЛА

Гетманенко Людмила Миколаївна
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Анотація. У статті розглядаються властивості бідотичного кола, яке визначається як коло, що дотикається до двох сторін трикутника та до описаного кола навколо цього трикутника. Задача, пов’язана з таким колом, вперше була запропонована на XX Міжнародній математичній олімпіаді (1976 р., Бухарест) для рівнобедреного трикутника. Однак пізніше було виявлено, що умова задачі є загальною для довільного трикутника. Вивчення цієї проблеми привернуло увагу математиків різних країн, і з'явилося чимало спроб доведення. У статті представлено новий підхід до розв'язку задачі за допомогою формул, що дозволяють спростити доказ, порівняно з іншими методами, наприклад, лемою Архімеда. Крім того, досліджуються окремі властивості бідотичного кола, що мають потенціал стати основою для задач підвищеної складності. Матеріали статті можуть зацікавити математиків, викладачів, студентів і всіх, хто цікавиться геометрією.
Ключові слова: бідотичне коло; внутрішній дотик кіл; інцентр трикутника; теорема косинусів; теорема “Трилисника”.

