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CIRCLE-ENHANCED TRAPEZOID: GEOMETRIC PROPERTIES AND THEOREMS

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Summary. *This paper explores the geometric properties that arise from the interaction between a trapezoid and a circle. Through a series of specific problems, it is demonstrated how incorporating a circle into the construction of a trapezoid gives rise to new, non-obvious statements that warrant independent theoretical investigation. In analogy with well-known results in triangle geometry—such as Simson's theorem, Euler's formula, and the nine-point circle—the combination of a trapezoid and a circle also reveals potential for the discovery of new theorems. The author examines a number of original geometric phenomena related to incircles and circumcircles of trapezoids, and provides proofs of their validity. The work aims to broaden the understanding of the circle as a tool for enriching geometric figures, particularly quadrilaterals.*
Keywords: *circle, circumcircle of a trapezoid; incircle of a trapezoid; triangle's excircle; right trapezoid; inscribed angles; central angles*

Recent studies emphasize the growing role of planimetry in developing students' mathematical proficiency, particularly through the use of digital technologies for visualization and problem-solving [2]. Within this educational context, investigating the geometric properties of trapezoids in relation to circles expands the toolkit for a deeper understanding of planar figures and their applications.

Let us consider the first problem, which is well known not only in school-level geometry but also in the preparatory stages of mathematics olympiads.

Problem 1. A circle of radius 1 cm is inscribed in the right trapezoid $ABCD$ with $BC \parallel AD$. The diagonals AC and BD intersect at point Q. Find the area of triangle CQD (see Figure 1).

Given:

$ABCD$ is a right trapezoid with $BC \parallel AD$, $\gamma = (O; R = OT)$ is a circle inscribed in the trapezoid with radius $R = OT =$1 cm.

Find:

S_{CQD}.

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Solution

1. Since $ABCD$ is a right trapezoid and $AB = 2R = 2$ cm, the triangle CQD is symmetric with triangle ABQ, and both have equal areas: $S_{ABQ} = S_{CQD} = S_{ABO}$;

2. The area of triangle ABO is given by:

$$S_{ABO} = \frac{1}{2} \cdot AB \cdot R \ (R \perp AB)$$

$$S_{ABO} = \frac{1}{2} \cdot 2 \cdot 1 = 1 \ (\text{cm}^2)$$

Thus,

$$S_{CQD} = 1 \ \text{cm}^2.$$

Answer: $1 \ \text{cm}^2$.

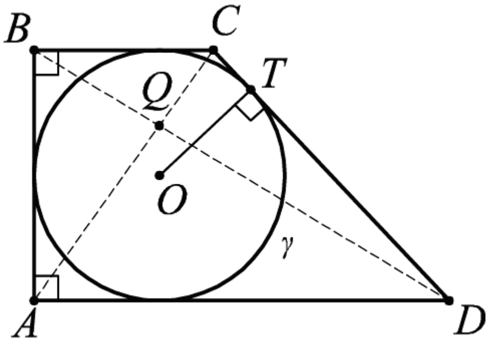

Fig. 1

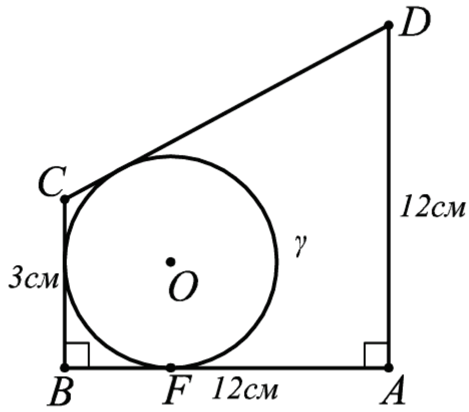

Fig. 2

<u>Problem 2.</u> In a right trapezoid $ABCD$ with bases BC and AD, and lateral side AB perpendicular to the bases, a circle is inscribed such that it is tangent to three sides: AB, BC, and CD (see Figure 2). Given: $AB = 12$ cm, $BC = 3$ cm, $AD = 12$ cm. Find the radius of the inscribed circle.

Given:

$ABCD$ is a right trapezoid with $BC \parallel AD$;
$AB = 12 \ cm$;
$BC = 3 \ cm$;
$AD = 12 \ cm$;
$\gamma = (O; OF = r_O)$ — the inscribed circle.

Find: r_O, the radius of the circle.

Solution

To solve this problem, recall the formula for the radius of an *excircle* in a triangle [4]:

For triangle CTB, the excircle touches the extension of side BC. Let p be its semiperimeter.

$TE_2 = TE_1 = p$

From triangle properties and symmetry (equal angles and congruent auxiliary triangles), we use the formula:

$$r_I = p \cdot tg\frac{\angle BTC}{2} = p \cdot \frac{r}{KT} = p \cdot \frac{r}{p-BC} = \frac{p \cdot r}{p-BC} = \frac{S}{p-BC},$$

$$r_I = \frac{S}{p-BC} \quad (1)$$

(see Figure 3) where S is the area of triangle TBC, and p is its semiperimeter.

To proceed, we enhance Figure 2 with auxiliary constructions (see Figure 4).

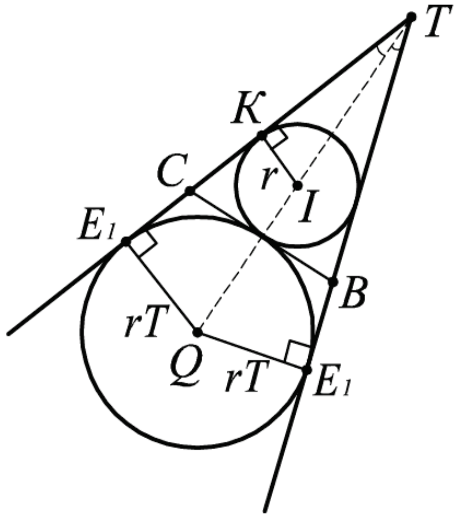

Fig. 3

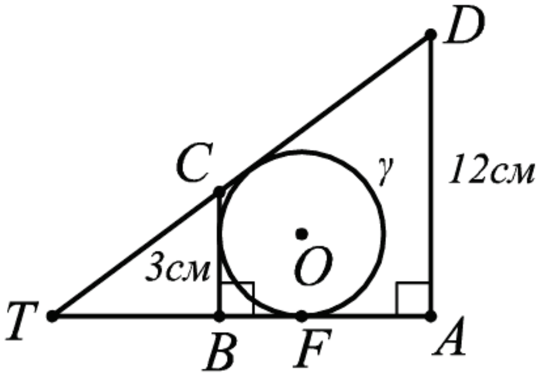

Fig. 4

Since the circle with center O is an excircle of triangle TBC, we apply formula (1):

$$OF = \frac{S_{TBC}}{p - BC}$$

or

$$r_O = \frac{S_{TBC}}{p - BC}.$$

Now consider similar triangles TCB and TDA. Using the similarity ratio:

$$\frac{BC}{AD} = \frac{TB}{TA}; \frac{3}{12} = \frac{TB}{TB + 12}; TB = 4 \text{ cm}.$$

The triangle TBC is a right triangle with legs 3 cm and 4 cm (i.e., a classic 3–4–5 triangle). Therefore:

- Area: $S_{TBC} = \frac{1}{2} \cdot 3 \cdot 4 = 6\ cm^2$,
- Semiperimeter: $p = \frac{3+4+5}{2} = 6\ cm$.

Substitute into the formula:

$$r_O = \frac{S_{TBC}}{p - BC} = \frac{6}{6 - 3} = 2 \text{ cm}.$$

Answer: $r_O = 2$ cm.

Problem 3. Let AD and BC be the bases of an isosceles trapezoid $ABCD$. Let P be the point of intersection of its diagonals. Prove that the circumcircles of triangles ABP and CDP intersect at the center of the circumcircle of trapezoid $ABCD$ [1].

Proof

Let O be the center of the circumcircle of trapezoid $ABCD$ (see Figure 5).

Angle $\angle CDB$ is an inscribed angle, and angle $\angle COB$ is the corresponding central angle.

By symmetry, the segment OP is the angle bisector of angle $\angle COB$.

Therefore, angles $\angle CDB$ and $\angle COP$ are equal.

It follows that quadrilateral $CDOP$ is cyclic (see Figure 6).

Proved.

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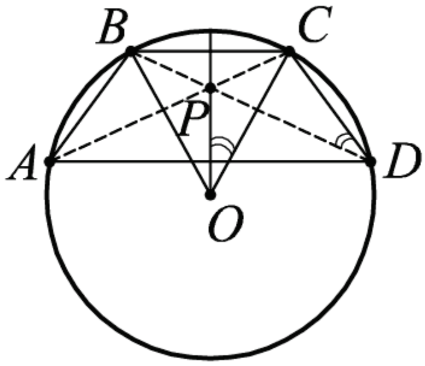

Fig. 5

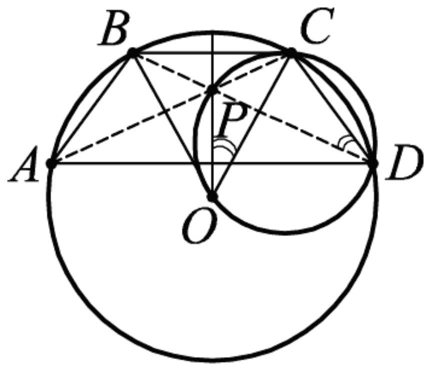

Fig. 6

Problem 4. Two trapezoids with correspondingly parallel sides are inscribed in the same circle. Prove that a diagonal of one of them is equal in length to a diagonal of the other.

Proof

Let angle $\angle MNE = \varphi$, then by the formula for a chord length in terms of the inscribed angle [3]:

$ME = 2 \cdot R \cdot \sin\varphi$,

where R is the radius of the circle circumscribed about the trapezoid.

But angle $\angle M_1N_1E_1$ is also equal to φ, so:

$$M_1E_1 = 2R \cdot \sin\varphi \text{ (see Figure 7).}$$

Thus,

$$ME = M_1E_1.$$

This completes the proof.

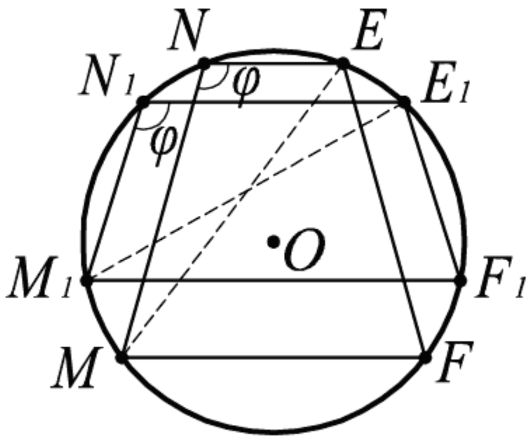

Fig. 7

Problem 5. The bases of a trapezoid, which is both inscribed in a circle and circumscribed about a circle, are m and n, where $m > n$. Find the distance between the centers of these two circles.

Solution

Let $\gamma_1$be the circle circumscribed about trapezoid $MNEF$, with center O and radius $R = MO$.

Let γ_2 be the circle inscribed in trapezoid $MNEF$, with center I and radius $r = IK$ (see Figure 8).

Then, $MF = m$, $NE = n$, hence $NK = \frac{n}{2}$, $MT = \frac{m}{2}$.

Draw the height EH of the trapezoid and denote it by h, where $h = 2r$ (see Figure 9).

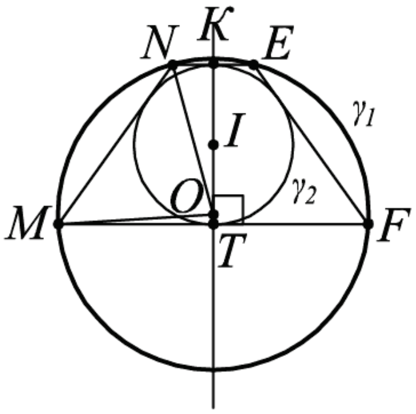

Fig. 8

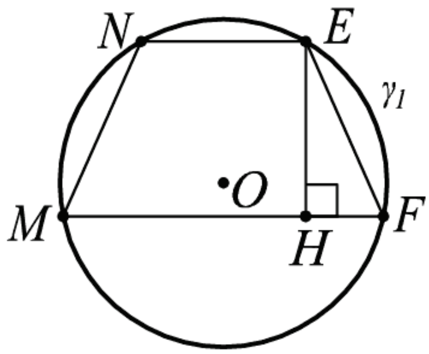

Fig. 9

Since the trapezoid $MNEF$ is inscribed in the circle γ_1, we have $MN = EF$.

Because it is also circumscribed about the circle γ_2 it follows that

$$MN + EF = MF + NE,$$

and so:

$$EF = \frac{MF+NE}{2} = \frac{m+n}{2}.$$

The height of the isosceles trapezoid is given by:

$$EH = \sqrt{(EF^2 - HF^2)} = \sqrt{(\frac{m+n}{2})^2 - (\frac{m-n}{2})^2} = \sqrt{mn},$$

and therefore,

$$r = \frac{\sqrt{mn}}{2} \qquad (2)$$

Let IK=x, where K is the midpoint of base ME.

From triangle MOT (see Figure 8), we have:

$$MO^2 = MT^2 + (KT - KO)^2,$$

or:

$$R^2 = \frac{m^2}{2} + (h - x)^2 \qquad (3)$$

From triangle NOK (see Figure 8), we also have:

$$ON^2 = NK^2 + OK^2$$

or:

$$R^2 = \frac{n^2}{2} + x^2 \qquad (4)$$

Equating expressions (3) and (4):

$$\frac{m^2}{4} + (h - x)^2 = \frac{h^2}{4} + x^2$$

Solving for x, we obtain:

$$x = \frac{m^2 - n^2 + 4h^2}{8h} \qquad (5)$$

Thus, the distance between the centers of circles γ_1 and γ_2 is:

$$OI = OK - IK = x - r = \frac{m^2 - n^2 + 4h^2}{8h} - \frac{\sqrt{mn}}{2}$$

Substituting $h = \sqrt{mn}$ into the expression:

$$OI = \frac{m^2 - n^2}{8\sqrt{mn}}$$

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Answer: $OI = \frac{m^2-n^2}{8\sqrt{mn}}$

Problem 6. Circles are constructed on the lateral sides of a trapezoid as diameters. Prove that the lengths of the tangent segments drawn from the intersection point of the trapezoid's diagonals to these circles are equal [5] (see Figure 10).

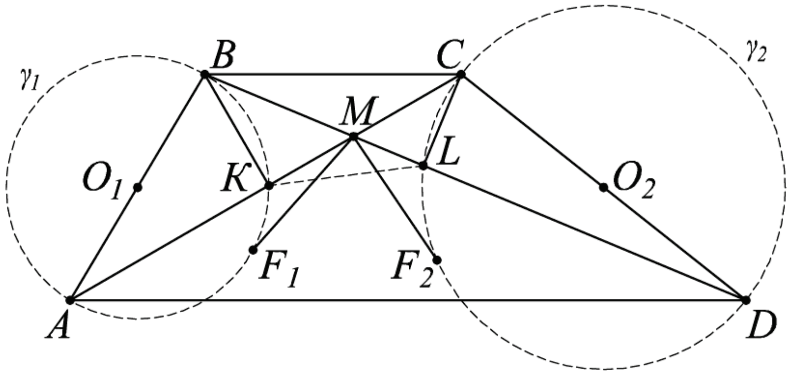

Fig. 10

Given:
$ABCD$ is a trapezoid with $BC||AD$;
$\gamma_1 = (O_1; R_1 = O_1B)$,
$\gamma_2 = (O_2; R_2 = O_2C)$,
$BD \cap AC = M$,

MF_1, and MF_2 are tangents drawn from point M to circles γ_1 and γ_2 respectively.

To prove:
$MF_1 = MF_2$

Proof
We note that:
$\angle BKC = \angle BKA = 90°$, $\angle CLB = \angle CLD = 90°$
Therefore, quadrilateral $BKLC$ is cyclic.
From this, we get:
$\angle CBL = \angle CKL$ (as inscribed angles subtending the same arc CL (see Figure 11).
Also:
$\angle CBL = \angle BDA$ (as alternate interior angles with $BC \parallel AD$, and BD a transversal.

Now consider triangles MKL and MDA.

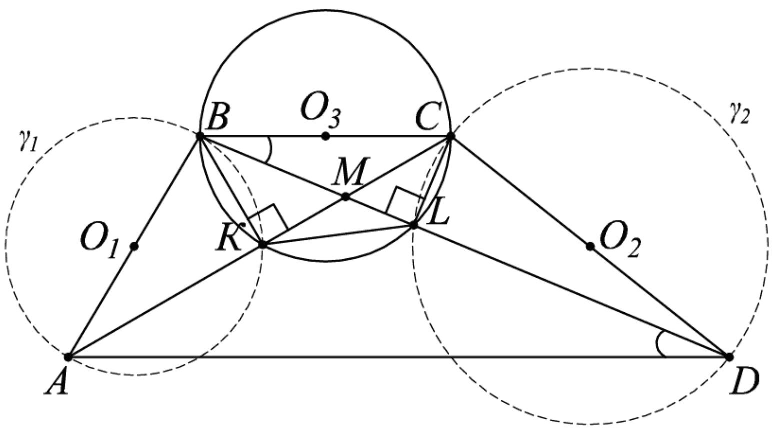

Fig. 11

From equal angles $\angle CBL = \angle CKL$, $\angle CBL = \angle BDA$, and the common angle $\angle M$, we conclude:

$$\Delta MKL \sim \Delta MDA$$

By the tangent-secant theorem (or the power of a point with respect to a circle), we have:

$$(MF_1)^2 = MK \cdot AM,\ \ (MF_2)^2 = ML \cdot MD$$

Due to similarity:

$$MK \cdot AM = ML \cdot MD \Longrightarrow MF_1 = MF_2$$

This completes the proof.

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ТРАПЕЦІЯ, ЗБАГАЧЕНА КОЛОМ: ГЕОМЕТРИЧНІ ВЛАСТИВОСТІ ТА ТЕОРЕМИ

Гетманенко Людмила Миколаївна
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Анотація. *У статті досліджено геометричні властивості, що виникають при взаємодії трапеції з колом. На прикладі конкретних задач показано, як включення кола у конструкцію трапеції породжує нові, неочевидні твердження, що заслуговують на окреме теоретичне осмислення. Аналогічно до відомих результатів у трикутнику, зокрема теореми Симсона, формули Ейлера та кола дев'яти точок, у випадку трапеції поєднання з колом також виявляє потенціал до відкриття нових теорем. Автор аналізує низку оригінальних геометричних явищ, пов'язаних із вписаними та описаними колами навколо трапеції, та пропонує докази їх істинності. Робота спрямована на розширення уявлень про коло як інструмент збагачення геометричних фігур, зокрема чотирикутників.*

Ключові слова: *коло, описане навколо трапеції; коло, вписане у трапецію; зовнівписане коло трикутника; прямокутна трапеція; вписані, центральні кути*

