Implementation of a Conceptual and Procedural Approach in University-Level Mathematics Education Using the Moodle LMS

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The paper discusses ways to enhance students' mathematical training, focusing on their ability to apply mathematics in practical, academic, or scientific contexts. We reviewed different scientific approaches, which either emphasise the development of conceptual understanding of mathematics or procedural knowledge. We concluded that university-level mathematics education should be based on the conceptual and procedural triad, which comprises conceptual understanding, procedural fluency, and problem-solving ability. We proposed teaching practices based on the educational strategy of Inquiry-based learning (IBL). These practices include (i) determining learning objectives to focus students' attention and motivate them to achieve them; (ii) solving problems with high cognitive demands that encourage students to search, research, solve problems, and make connections; (iii) engaging students in productive work on knowledge construction and fostering meaningful mathematical discourse; (iv) procedural fluency training based on conceptual understanding. The effectiveness of this approach has been tested experimentally. Special emphasis was placed on implementing an integrated conceptual and procedural approach using the Moodle LMS. This digital platform provided a flexible and convenient blend of various educational activities, which is especially beneficial for blended learning models. Moodle facilitated the organisation of ongoing interaction between lecturers and students, supported autonomous learning, and enabled feedback, all of which contributed to the overall effectiveness of mathematics education.

Keywords: integrated conceptual and procedural approach; mathematical competence; students' mathematical training; university-level mathematics education; technology; Moodle LMS.

1. INTRODUCTION

The modern educational paradigm involves a competency-based approach to education. The main goal of such competency-based education is to develop relevant competencies in a specific subject area. In mathematical education, in particular, this involves the development of mathematical competence, defined as the 'ability to understand, judge, do, and use mathematics in a variety of intra- and extra-mathematical contexts and situations in which mathematics plays or could play a role' (Niss, 2003). According to this widely cited interpretation of mathematical competence, as well as an analysis of its component competencies (Niss & Højgaard, 2019), it can be concluded that mathematical competence necessarily entails a conceptual understanding of mathematical concepts, facts, methods, procedural fluency, and, based on these, the ability to solve problems. This conceptual and procedural triad is a source that nourishes and a foundation upon which mathematical competence firmly stands, allowing for the practical application of mathematics to solve various purely mathematical and non-mathematical problems.

The role of conceptual and procedural knowledge in mathematics education has been the subject of several studies. Specifically, there is debate about which learning environment is more effective in teaching and learning mathematics – one that emphasises conceptual understanding (e.g. Crooks & Alibali, 2014; Kholid et al., 2021; Niemi, 1996 etc.) or one that prioritises the development of sustainable procedural skills. (e.g. Selden & Selden, 2011; Star, 2005 etc.).

Those favouring a conceptual learning environment, where the development of concepts precedes the development of skills, argued that understanding concepts is more important and that computers can successfully perform various computational procedures. On the other hand, proponents of the procedural approach in teaching and learning mathematics criticised the conceptualists for focusing too much on developing concepts and neglecting the practical application of mathematical methods. They argued that it's more beneficial to have an overview of the concepts and focus on explaining relevant algorithms and procedures while training practical skills.

Against this backdrop of disputes, the view on the fallacy of the conceptual-procedural dichotomy deserves attention; according to this perspective, conceptual and procedural approaches in mathematics education are closely interrelated and complement each other (e.g. Kieran, 2013; Wu, 1999, etc.).

Piaget emphasised this because he believed that conceptual understanding and procedural knowledge are integral components of a single cognitive system (Ginsberg & Opper, 1988). However, simply acquiring conceptual or procedural knowledge is not the ultimate goal of learning mathematics. The aim is to cultivate mathematical competence, which means being able to apply mathematics in practical, academic, or scientific contexts. Therefore, finding a learning model that can foster students' conceptual understanding of mathematics without neglecting procedural knowledge is essential.

Digital technologies, especially distance learning platforms, have significant potential to address educational challenges. Moodle, a widely used educational platform, can integrate tools for visualisation, interactive training, and feedback, which helps develop both conceptual understanding and procedural knowledge simultaneously (Mlotshwa et al., 2020; Pahrudin et al., 2021; Psycharis et al., 2013). However, the effectiveness of these technologies largely depends on the pedagogical design and the teacher's role (Dorner & Ableitinger, 2022). Therefore, further research focused on substantiating and implementing approaches for the use of Moodle, aimed at fostering students' conceptual understanding of mathematics and their procedural knowledge, is particularly important.

Thus, our study aims to address two research questions:

(i) Why is it essential to integrate conceptual and procedural approaches in university-level mathematics education?
(ii) What effect did the integrated conceptual and procedural approach to university-level mathematics education using Moodle have on students' mathematical training outcomes?

2. LITERATURE REVIEW

2.1 Conceptual understanding and procedural knowledge in mathematics

Contemporary educational literature significantly focuses on procedural knowledge and conceptual understanding of mathematics. Supporters of procedural knowledge and related learning approaches emphasise the practicality of using a series of steps or algorithms to solve a mathematical problem (e.g. Selden & Selden, 2011; Star, 2005 etc.). Conversely, approaches prioritising conceptual understanding of mathematics aim to provide a deeper insight into applying a known procedure to a new situation. Furthermore, conceptual understanding fosters stronger connections between mathematical concepts and ideas (e.g. Crooks & Alibali, 2014; Kholid et al., 2021; Niemi, 1996, etc.).

Numerous scientists have identified issues of procedural teaching and learning mathematics (DeCaro, 2016; Richland et al., 2012 etc.). Students memorise procedures for solving mathematical problems without truly understanding why these procedures work. Consequently, students often apply these procedures thoughtlessly to solve new problems. Essentially, students view mathematics as a set of rules and procedures to be memorised while lacking the fundamental concepts necessary for reasoning about mathematics.

Researchers have different opinions about interpreting the terms 'conceptual' and 'procedural' and how to balance them when learning and teaching mathematics (Bossé & Bahr, 2008). Another pedagogical issue is that not all teachers are prepared to teach students in a way that helps them understand the conceptual aspects of mathematics. For example, a study by Stovner and Klette (2022) examined 47 Norwegian mathematics teachers and found that they primarily provided feedback on procedural skills, while giving less attention to conceptual feedback, which focuses on mathematical concepts and their relationships. Gándara and Contreras (2009) observed that many mathematics teachers struggle to balance teaching methods that emphasise conceptual understanding and procedural knowledge.

The relationships between conceptual understanding and procedural knowledge are often bidirectional, with procedural knowledge often supporting improvements in conceptual understanding and vice versa (Rittle-Johnson et al., 2015). Phuong (2019) considered integrating procedural and conceptual knowledge in mathematics education. The author noted that students' ability to integrate procedural and conceptual knowledge when solving mathematical problems was low. In 2001, it was demonstrated that mathematical proficiency consists of five components, or strands, including: conceptual understanding – comprehension of mathematical concepts, operations, and relations and procedural fluency – skill in carrying out procedures flexibly, accurately, efficiently, and appropriately (National Research Council, 2001). It is recommended to engage students in specially organised training that promotes the integration of procedural and conceptual knowledge.

Thus, contemporary scientific theory supports implementing an integrated conceptual and procedural approach in teaching and learning mathematics. However, there is a lack of actual practices for implementing this approach and confirming its effectiveness.

2.2 Experience using technology to enhance students' conceptual understanding of mathematics and procedural knowledge

The use of technologies in mathematics education contributes to a deeper conceptual understanding through the visualisation and exploration of ideas, as well as strengthening procedural knowledge through interactive training and adaptive feedback (Ghuniamat, 2024; Gün Sahin and Kırmızıgül, 2023; Zulnaidi et al., 2017 etc.). Thus, Gün Sahin and Kırmızıgül (2023) discussed their experiences with using technology to foster both procedural and conceptual understanding in students. They specifically highlighted techniques for using interactive presentations, videos, and digital tools such as MindMeister, Padlet, Google Forms, and Canva. Their findings indicated that learning in an electronic environment led to a high success rate (76%) in completing tasks that required the application of procedural knowledge.

A study by Zulnaidi et al. (2017) revealed that using GeoGebra enhanced students' conceptual understanding of the topic of "Functions". This improvement was attributed to the software's ability to clarify the core concepts and connect them directly to everyday life.

At the same time, the leading teacher's role and the pedagogical strategies they employ were highlighted since the use of digital technologies in itself does not ensure effective mathematics teaching. Specifically, a study involving 455 Austrian students found no correlation between the frequency of technology use, such as GeoGebra or graphing calculators, and the results of tasks requiring the application of procedural knowledge (Dorner & Ableitinger, 2022).

Several studies highlighted the potential of Moodle as a tool for developing both conceptual understanding and procedural knowledge in mathematics (Heba et al., 2014; Mlotshwa et al., 2020; Pahrudin et al., 2021; Psycharis et al., 2013 etc.). Notably, it has been demonstrated that learning with Moodle has a significant positive impact on developing students' conceptual understanding of algebra (Pahrudin et al., 2021). Mlotshwa et al. (2020) shared their experience using Moodle to foster conceptual understanding of the topic of "Functions". In their hybrid learning model, they found that Moodle's features played a crucial role in improving understanding of mathematical functions. Furthermore, using Moodle was found to increase students' responsibility for their learning and to enhance their overall learning outcomes.

Thus, current research confirms that technologies, particularly Moodle, can aid in developing students' conceptual understanding and procedural knowledge in mathematics. However, no study has yet demonstrated the potential of using Moodle to simultaneously implement both conceptual and procedural approaches in mathematics education at the university level. This gap presents promising opportunities for further scientific research.

3. METHODS

3.1 Participants

The experiment took place between September 2023 and June 2024. It involved two universities in Ukraine: Sumy State Pedagogical University named after A.S. Makarenko [control group U1 (N=8)] and Boris Grinchenko Kyiv Metropolitan University [experimental group U2 (N=11)]. The limited number of students in both samples can be attributed to our intention to conduct a pilot study. Positive responses to the research questions will serve as the foundation for a larger experiment involving more participants in the future. Approval for the experiment was obtained at Loughborough University, where one of the researchers worked. Both groups of students participated in blended learning through the Moodle LMS. The experiment involved two groups of first-year students in the "Mathematics" program. The lecturers who taught the course of mathematical analysis have extensive experience teaching this course. The educator in the experimental group has experience using a conceptual approach in their teaching practice (Astafieva et al., 2023), primarily through the PLATINUM Project (Astafieva et al., 2021; Rogovchenko et al., 2021). Two mathematics teachers also participated in the study as observers and experts.

3.2 Content of learning mathematics

Students studied mathematical analysis in their first year at both universities. The course covered sets, numerical sequences and their limits, functions of one real variable, limits and continuity of functions, differential and integral calculus of functions, and series. Training in the experimental group was conducted using an integrated conceptual and procedural approach. The control group followed a traditional study approach, primarily using informative-reproductive teaching. In this method, the lecturer explains theoretical facts, formulas, and their proofs, while providing standard methods for solving typical problems. This approach tends to focus more on training specific procedures rather than fostering a deep understanding of the underlying processes.

3.3 Instruments

3.3.1 Comparison of student groups at the beginning of the experiment

At the beginning of the experiment, we needed to determine if groups U1 and U2 had the same level of secondary school mathematical training. To achieve this, we initially compared the results of the national multi-subject test in mathematics (NMT) for students in groups U1 and U2 (The Ukrainian Centre for Educational Quality Assessment, 2024). The Mann-Whitney U test was used to compare the results of NMT.

Next, we administered a diagnostic test in Moodle to assess the students' knowledge of algebra, geometry, and the fundamentals of mathematical analysis (see Appendix 1). The test consisted of 10 tasks that assessed students' conceptual understanding of mathematics, procedural fluency, and problem-solving abilities. The tasks were designed to be accessible to an average school graduate and based on the authors' university teaching experience.

3 out of 10 test tasks were identified by the authors and two independent experts as conceptual, meaning their solutions required mainly conceptual understanding. 3 were identified as procedural, indicating that good procedural knowledge was sufficient for their solutions. 2 tasks were identified as both conceptual and procedural, requiring conceptual understanding and procedural knowledge to the same extent. 2 tasks were mathematical modelling tasks with real-life content. A level scheme was developed to determine the type of mathematical tasks (see Appendix 2). The task evaluation results by the authors and experts are provided in Appendix 3.

The performance of each task was assessed on a four-point scale (see Appendix 4). The 90-minute time allocated for task completion was considered sufficient, so lack of time was not an acceptable reason for not finishing the test.

3.3.2 Comparison of student groups at the end of the experiment

After conducting experiments, we compared the results of students' mathematical training groups U1 and U2 by analysing their final diagnostic test and the end-of-term exam in mathematical analysis.

The students completed the final test in Moodle, which consisted of 15 tasks covering elementary mathematics and mathematical analysis. The tasks were divided into three sections, but the names of the sections were not disclosed to the students.

(i) Understanding concepts, facts, and methods (Section I).
(ii) Procedural fluency (Section II).
(iii) Problem-solving and mathematical modelling (Section III).

The study's authors designed the test, and its tasks were assessed on a four-point scale, similar to the initial assessment.

The respondents were required to provide explanations for each task and offer complete solutions. They were encouraged to use different solution methods and types of presentation, such as tables, diagrams, graphs, and symbolic records. Students were asked to complete some individual test tasks using only pen and paper without any digital tools and upload a photo of the solution to Moodle. The time allocated for completion was 120 minutes. The decision to prohibit using digital tools during test tasks was based on the objective of evaluating students' depth of understanding of mathematical concepts, their interrelationships, and their ability to execute procedures with meaning and skill. The tasks can be found in Appendix 5.

The idea behind comparing the results of the mathematical analysis exam was to assess the effectiveness of an integrated conceptual and procedural approach to the mathematical preparation of students.

3.3.3 Questions for experts and observers

Experts who served as observers were involved at different stages throughout the experiment. Their roles included:

(i) Ensuring the diagnostic materials are aligned with the research objectives.
(ii) Verifying that the content of educational sessions reflected an integrated conceptual and procedural approach.
(iii) Participating in discussions regarding interim and final research results.

Throughout the research process, interviews were conducted with observers/experts to ensure the accurate implementation of the experiment and to discuss the research results.

4. RESULTS

We will seek an answer to the first research question, "Why is it essential to integrate conceptual and procedural approaches in university-level mathematics education?"

4.1 The conceptual and procedural triad is the foundation of mathematical competence

Conceptual understanding in mathematics refers to a profound comprehension of mathematical concepts, principles, and their interconnections. It goes beyond memorising facts and ideas; it involves understanding the relationships between them, the reasons behind mathematical formulas and rules, and how they can be used in various situations. It includes:

1. The ability to identify, label, provide examples and counterexamples, apply different interpretations, and justify research methods and tools.
2. Understanding the connections between concepts across different areas of mathematics.
3. Understanding the core ideas and the purpose of a specific proof, mathematical method, its application, and any relevant limitations.

Procedural fluency refers to the effective execution of procedures, such as calculations, transformations, and algorithms. It includes the following:

1. Logical thinking and the ability to make or refute conclusions.
2. Knowing when and how to use specific procedures and choose appropriate mathematical methods and tools.
3. Performing procedures accurately, flexibly and confidently.

Problem-solving ability is a multi-stage skill that includes the following components:

1. Problem analysis: interpreting the conditions, identifying essential data, and determining the type of problem.
2. Strategy development: selecting suitable methods and creating a solution plan.
3. Implementation: applying the relevant mathematical procedures to reach a solution.
4. Evaluation: verifying the accuracy of the procedures and assessing the results for realism and consistency with the problem conditions.
5. Generalisation: clarifying the theoretical foundations and relating the results to existing knowledge.

Hence, the capacity to solve problems encompasses a range of interconnected skills and knowledge that form a cohesive foundation for effectively solving mathematical problems. Successful problem-solving depends on conceptual understanding and procedural fluency and is activated by conceptual and procedural knowledge (see Fig. 1). Solving problems deepens conceptual understanding and 'hones' procedural skills.

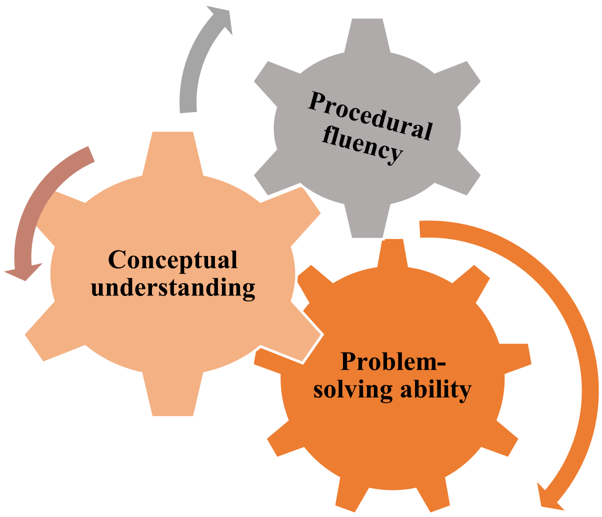

Figure 1. Dynamic connection of the links of the conceptual and procedural triad

4.2 The essence of an integrated conceptual and procedural approach to learning mathematics

Next, we will explore the importance of integrating conceptual and procedural approaches in teaching and learning mathematics based on the findings of previous studies (Ho, 2020; Phuong, 2020; Phuong, 2019; Rittle-Johnson et al., 2015; Kieran, 2013; Zahner et al., 2012; Wu, 1999; Hiebert, 1986). It is evident that neither conceptual nor procedural knowledge is an end in itself when learning mathematics. The goal is a developed mathematical competence, which involves applying mathematics in practical, academic, or scientific activities. The issue of how to best achieve this objective remains relevant: whether to prioritise the development of a conceptual understanding of mathematics or to emphasise the development of procedural knowledge.

An integrated approach involves:

1. Conceptual component: Understanding the principles, structures, and logic of mathematical processes, and applying conceptual frameworks to investigate, explain, and prove.
2. Procedural component: Practising and automating algorithms and methods to solve problems effectively.
3. Integration of concepts and procedures: Developing a conceptual understanding that includes a clear study of procedures as an integral and organic part of the conceptual approach.
4. Problem solving: Selecting strategies and applying knowledge to analyse, solve, and generalise problems.

The sequence in which procedures and concepts are learned in an integrated conceptual and procedural approach is crucial because "teaching first for conceptual knowledge leads to the acquisition of procedural knowledge, but the converse is not true" (Brown et al., 2002).

4.3 Moodle's capacity to implement an integrated conceptual and procedural approach to learning mathematics

To ensure the holistic development of a student's mathematical competence, we view it as a combination of three interrelated components: conceptual understanding, procedural fluency, and problem-solving ability. This requires the use of appropriate teaching strategies and digital tools that support the gradual and interconnected growth of each component. Moodle is a highly effective platform for blended learning, a modern learning management system that enables educators to create a multi-level, dynamic, and differentiated educational environment.

Using Moodle facilitates a flexible learning design, focusing on both mastering key mathematical concepts and developing procedural knowledge and problem-solving strategies. Theoretical material can be presented through interactive lectures, multimedia explanations, or hypertext modules that support a deeper understanding of concepts and their connections. To develop procedural knowledge, it is advisable to use automated tasks, including both open-ended and multiple-choice questions. Additionally, providing dynamic feedback helps encourage self-assessment and facilitates error correction.

Research-oriented teaching methods play a crucial role in this approach and are effectively supported by Moodle tools. Activities such as forum discussions, collaborative projects, and analytical tasks, combined with visualisation tools like Miro or GeoGebra, help students develop critical thinking, argumentation skills, mathematical modelling, and a creative approach to problem-solving. Additionally, lecturers can conduct flexible formative assessments to monitor students' progress at every stage, from their initial understanding to their ability to explore mathematical concepts independently.

An example of implementing this approach is an e-learning course on mathematical analysis for students in the experimental group U2, designed in Moodle. Its structure includes thematic modules containing:

- *A list of expected learning outcomes*, detailing what students should understand, know, and be able to do.
- *Summaries of lectures* formatted for Moodle, featuring varied learning pathways, transitions between pages, additional content clusters, hidden text, interactive visualisations, and pages with test questions or tasks for

automatic grading after logically completed segments of text.

- *Practical modules* with training tasks for practising standard procedures, organised in Moodle Assignment format.
- *Mini-research and open-ended tasks* that require creative problem-solving approaches, formatted in a Wiki style, for example.
- *Discussion forums* for analysing solutions, identifying common errors, and exploring different strategy options.
- *Individual assignments* formatted in Moodle Assignment style, incorporating elements of self-assessment and peer assessment.
- *Self-testing modules* that provide instant feedback, including the use of STACK.
- *Questions and tasks* are designed in Moodle Assignment or Quiz format to evaluate and assess knowledge and skills.

Thus, Moodle tools provide the technical and methodological basis for the full implementation of an integrated conceptual and procedural approach. They enable the effective combination of various learning activities aimed at developing a deep understanding, mathematical techniques, and the ability to think critically while solving problems.

4.4 Implementation of an integrated conceptual and procedural approach in Moodle

The teacher's educational strategy was Inquiry-Based Learning (IBL) when the educational environment encourages students to act actively, to construct their own knowledge through the methods typical of professional mathematicians (Gómez Chacón et al., 2021).

The conceptual and procedural approach was based on the principle: "First understanding, then procedures." Let's discuss some of the most effective practices that have led to positive educational results:

(i) Determining learning objectives to focus students' attention and motivate them to achieve them.
2. Solving problems with high cognitive demands that encourage students to search, research, solve problems, and make connections.
3. Engaging students in productive work on knowledge construction and fostering meaningful mathematical discourse.
4. Procedural fluency training based on conceptual understanding.

Let's delve into each teaching practice in more detail.

4.4.1 Determining learning objectives to focus students' attention and motivate them to achieve them

When determining learning objectives for a lecture or practical session, the focus should be on what students will learn through active engagement in the learning process. To ensure that students are motivated and mobilised to achieve educational objectives, they should participate in defining these objectives.

We will provide an example of a collaborative formulation of learning objectives during one of the U2 group lectures, which was conducted remotely using Zoom. The lecture topic was "Convergence test for a positive numerical series". The lecturer invited the students to share their expectations for the benefits of this lecture, leading to a 5-minute collaborative discussion on Padlet about the learning objectives (see Fig. 2).

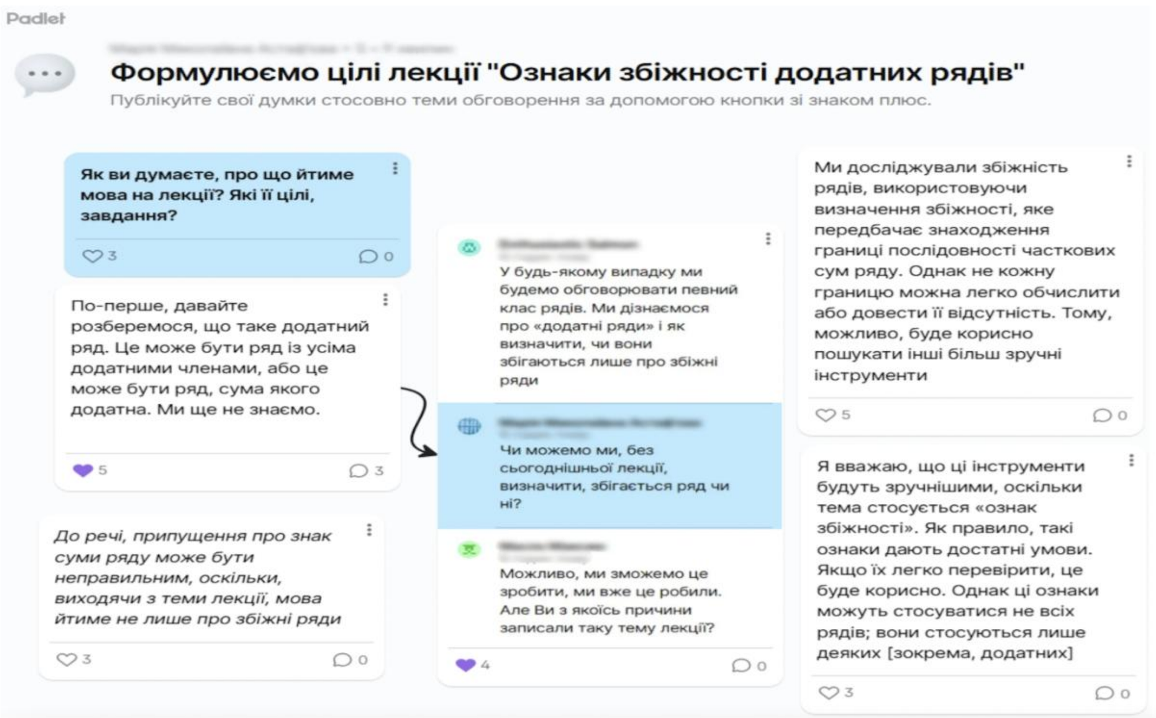

Figure 2. Discussion of the lecture objectives; questions posed by the lecturer are presented on a blue background (in Ukrainian)

Here is the English translation of the discussion that was published on Padlet:

> *"Firstly, let's find out what a positive series is. It could be a series with all positive terms, or it could be a series whose sum is positive. We don't know yet."*
>
> *"The assumption about the sign of the sum of the series may be incorrect because, based on the lecture topic, it will not only be about convergent series."*
>
> "[As if summarising] *In any case, we will be discussing a particular class of series. We will learn about 'positive series' and how to determine whether they converge."*

Then, the lecturer, purposefully [to deepen the awareness of the issue and thereby strengthen motivation] joins the discussion by asking, "Can we determine if a series converges without today's lecture?" Here are the students' responses:

> *"Maybe we can do it, we've already done it. But did you* [to the lecturer] *write down such a lecture topic for any reason?"*
>
> *"We examined the convergence of series by using the definition of convergence, which involves finding the limit of the sequence of partial sums of the series. However, not every limit can be easily determined or proven not to exist. Therefore, it may be helpful to look for other, more convenient tools."*
>
> *"I believe these tools will be more convenient because the topic pertains to the 'test of convergence.' Typically, such a test represents sufficient conditions. If they are easy to verify, that would be beneficial. However, these signs may not apply to all series; they only apply to some* [specifically the positive ones]*."*

The lecturer and, at their request, four students compiled a summary of the discussion on Padlet in the form of a table with columns: (i) Questions; (ii) Clarification of the problem; (iii) What do we want to find out? (see Fig. 3).

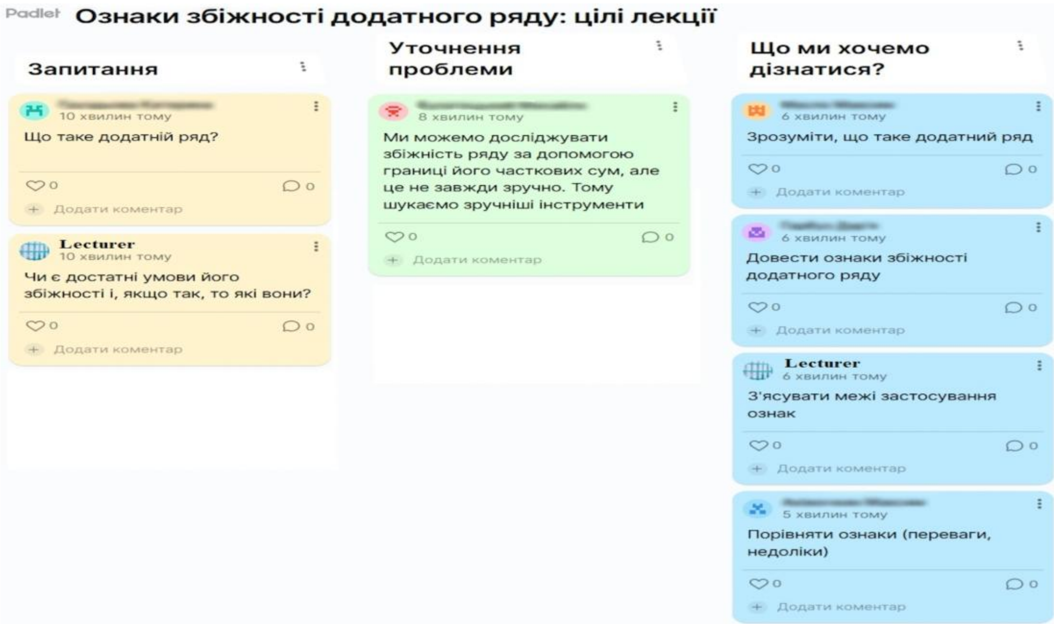

Figure 3. Screenshot of lecture objectives created on Padlet (Ukrainian)

Learning objectives have become a motivating factor for students, fostering their interest in the learning course. This active involvement in the educational process empowered each student to play an active role in the lecture.

4.4.2 Solving problems with high cognitive demands that encourage students to search, research, solve problems, and make connections

When we refer to the high cognitive demands of a task, we are not considering whether the task has real-life significance or is purely a mathematical problem. We are also not referring to the conditions under which the task will be solved, whether independently or collectively. We are explicitly emphasising the opportunities this task provides students who solve it. Namely, opportunities that help develop their mathematical thinking and support cognitive and mathematical activity. It is essential to note that problems that can be solved in various ways, as well as those associated with search and research, have a more significant developmental potential. The cognitive requirements of 'developmental' tasks should encompass justifying and explaining the procedures used and employing strategies commonly used by mathematicians, such as observation, experimentation, making conjectures, providing justifications, interpretation, etc.

Here is an example of a similar problem that was proposed to students during one of the practical classes on the topic "Series":

"Determine the convergence of the series

$$2+\left(\frac{5}{4}-\frac{7}{8}\right)+\left(\frac{10}{9}-\frac{26}{27}\right)+\ldots+\left(\frac{n^2+1}{n^2}-\frac{n^3-1}{n^3}\right)+\cdots,$$

and also determine the convergence of the series formed by removing the parentheses. Does the result you obtained contradict the associative property of a convergent series?"

This task aims not only to train and test procedural knowledge of analysing the convergence of series but also to deepen conceptual understanding. That is, it was designed to establish a connection with previously studied theoretical material and comprehend the essence of the relevant theorems.

In Figure 4, the solution provided by one of the students is shown. It shows that the student proved the convergence of the first series by comparing it with the convergent series of inverse squares and demonstrated the divergence of the second series using the necessary convergence condition. In response to the lecturer's conceptual question (item 3 of the figure), the student stated,

> "No, it does not contradict. The theorem on the associative property of a series says that if the series is convergent, then it has this property. Nothing is said about the divergent series."

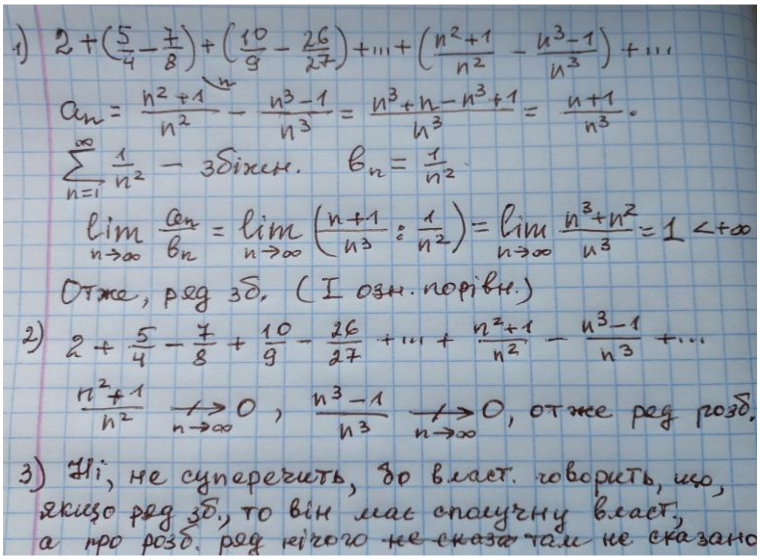

Figure 4. Solving the problem by the student

The task was not only for students to solve it, but also for them to mutually evaluate each other's solutions, which is undoubtedly helpful for both the development of conceptual understanding and training in critical thinking and argumentation skills. The Workshop activity tool in the Moodle system was used to organise this activity. This module provides a transparent and convenient environment for mutual evaluation, while also allowing the lecturer to monitor the group's progress, provide consultations, answer questions, and coordinate student activities, taking into account their individual characteristics and needs. In addition, the module promotes active interaction between participants, providing feedback and the opportunity to exchange comments, impressions and reviews.

4.4.3 Engaging students in productive work on knowledge construction and fostering meaningful mathematical discourse

A necessary prerequisite for studying mathematics is students' active involvement in mastering mathematical knowledge, or more precisely, their construction. Without this involvement, they will not fully understand mathematics, and their learning ability will be severely limited. Research has shown that active mathematics learning (as opposed to passively informative) positively impacts educational outcomes (Freeman et al., 2014). Active learning strategies are a mechanism for developing a conceptual understanding of mathematical structures (the basis of qualitative conceptual and procedural knowledge), creative thinking, search-research competencies and meta-skills. One effective strategy for active learning is IBL, which was used to teach mathematical analysis in U2. The involvement of students in active learning followed the scheme outlined in Figure 5.

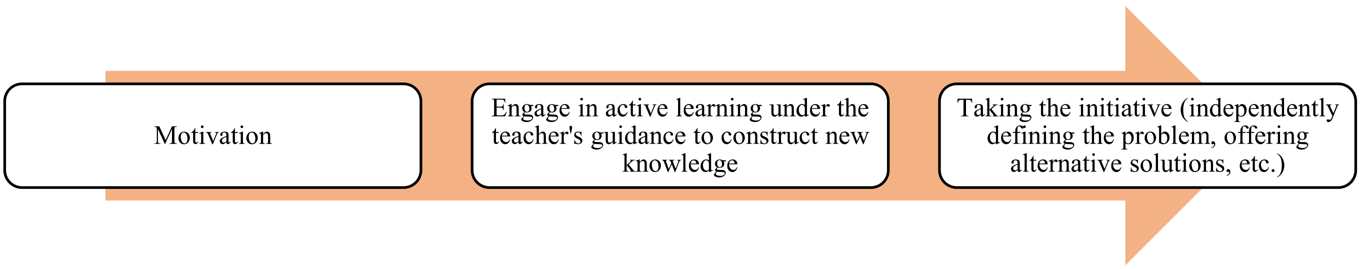

Figure 5. Scheme of involving students in active learning

In the paper, for brevity, we direct the reader to the description of portions of two lectures on mathematical analysis (Astafieva, 2021; Rogovchenko, 2021) delivered by the same lecturer to first-year mathematics students. The teacher involved students in active learning by posing purposeful inquiry questions to help them form a hypothesis about the absolute convergence of a series and develop the concept of the Riemann integral. Students were given tasks within their 'zone of proximal development' (Vygotsky, 1987) and received support to help them achieve positive results. The students' progress in independently formulating research problems and their readiness to work towards solutions is demonstrated in their use of the "Forum" in LMS Moodle (Astafieva, 2021).

Based on the experts' observations, the teacher assigned tasks to students to practice specific procedural skills and also gave them problems that required conceptual understanding, research efforts, creative thinking, and unconventional approaches. In the following lesson, students' solutions to these problems were usually discussed, which helped them learn how to assess their and other students' ideas and arguments. These discussions often led to the discovery of new problems and served as a starting point for gaining new knowledge.

An example of a task that aimed to deepen conceptual understanding and develop qualitative procedural knowledge during the study of the topic "Limit of a numerical sequence" was the following question:

> "How would you explain to a person unfamiliar with the concept of limit and not very involved with mathematics, what the limit of a numerical sequence is?"

Students were given time to familiarise themselves with the task in advance. During the next lesson, they presented their explanations and collectively discussed them. The teacher emphasised the importance of this task, saying, "The concept of limit is known to be one of the most challenging for first-year students, as it marks a significant leap from elementary to higher mathematics. They have difficulty with the formal definition of a limit because its structure is significantly different from the definitions they learned in school. Therefore, it is crucial for them to grasp the essence of this concept intuitively before delving into a strict mathematical formulation of the definition. I believe that explaining something to someone else is the best way for students to understand it themselves."

Students rigorously analysed each other's explanations during discussions, assessing their strengths and weaknesses. Every explanation made by the students had to be supported with strong arguments to be considered valid. The teacher then seamlessly transitioned the discussion to a detailed analysis of the precise definition of the limit of the numerical sequence, both verbally and symbolically. This thorough analysis aimed to ensure a deep understanding of the definition and eliminate any difficulties in its use and memorisation. For instance, the students explored the role of each phrase or symbol in the definition and examined potential outcomes if they were replaced with others.

In the context of online learning, it's essential to note that when students assumed specific temporary roles, such as tutor, expert, or opponent, the lecturer of the experimental group U2 effectively utilised the Moodle Wiki tool. This tool allows for the creation of collaborative pages that can be edited and commented on by other users granted access by the lecturer. Unlike the Glossary tool, the editing history is stored, enabling the lecturer to track the development of each student's understanding of certain mathematical concepts as well as the evolution of their skills and abilities. This information helped the lecturer make necessary adjustments to their teaching approach. Astafieva et al. (2023) discussed an example of how the lecturer implemented the Wiki tool in practice.

The effectiveness of this approach in developing both a conceptual understanding of complex mathematical concepts and procedural skills based on understanding, rather than mechanical memorisation, is confirmed by the example provided above. A month after studying the topic "Limits of Numerical Sequences," lecturers asked students from both groups, U1 and U2, to answer two questions:

1. The number a is the limit of the numerical sequence $\{x_n\}$. What does this mean?
2. Does the sequence given by the formula $x_n = (-1)^n$ have a limit? Please explain your answer.

The results were as follows: all 10 students in U2 correctly formulated the definition of the limit of a numerical sequence. Additionally, 8 of them not only provided a mental formulation but also expressed it using mathematical symbols and offered a geometric interpretation. They also answered the second question correctly and argued their answers.

In contrast, only one student in U1 correctly defined the limit of a sequence. 5 students attempted to do so but made various mistakes that indicated a misunderstanding of the concept itself. Furthermore, only 6 students provided the correct answer to the second question, but they did not justify their responses.

4.4.4 Procedural fluency training based on conceptual understanding

An integrated conceptual and procedural approach to learning mathematics considers explicit learning of procedures as a natural, organic part of conceptual learning. Mechanical memorising procedures without comprehending the related mathematical concepts, understanding when and why to apply them, and recognising their limitations will hinder the effective application of mathematics to solve new and complex problems.

The idea that focusing on understanding takes away time from training procedural skills is unfounded. Investing time in developing a conceptual understanding, including understanding the nature of the procedure itself, will have significant benefits. When a student understands not only how to perform a procedure but also why it works in a given situation, why it is performed in a certain way, and its limitations, the student will be able to apply the procedure correctly and effectively.

As for the importance of repeatedly applying certain procedures to achieve automaticity in their use, conceptual understanding not only does not hinder this process but actually facilitates it.

Practising mental and semi-mental problem-solving can enhance one's fluency in performing mathematical procedures, especially when grounded in a solid understanding of the concepts. Semi-mental problem-solving is a calculation method that involves performing some steps mentally, without detailed notes. This approach reduces the amount of written work while ensuring clarity and control over the calculations. It also fosters mathematical thinking and develops procedural skills, allowing you to train the fluency of performing procedures, which will undoubtedly be more accessible if rooted in understanding.

An observer described a seminar focusing on the topic "Calculation of an indefinite integral by substituting a variable" in U2 as follows, "We observed a seminar that was conducted remotely using the Moodle LMS. In previous classes, students studied the table of primitives and the theorem on the invariance of the indefinite integral formula. In this lesson, the lecturer planned to train students' ability to convert indefinite integrals into tabular integrals using a specific transformation

$$\int f(x)g(x)dx = k\int f(x)d\big(f(x)\big).$$

The lecturer transformed the first 15 minutes of the session into an engaging and effective competition, where students orally solved problems to find integrals of a specific type, for example $\int \frac{x^2dx}{x^6+4}$. For this, the lecturer used Moodle activities Assignment with built-in Miro Online Whiteboard (see Fig. 6). On this board, 10 small personal zones were created for each student, where they needed to write indefinite integrals. The students were required to complete this task quickly, without conducting detailed calculations. During this activity, the students managed to calculate 68 primitives. How long would it take to complete and write down in details such a volume of routine work in a notebook? [emotionally]. Aside from the beneficial repetition of table integrals and basic skills training, this 'warm-up' provided the students with positive emotions that contributed to their further work. It also allowed the lecturer to see each student's answers and mistakes simultaneously, enabling timely corrections."

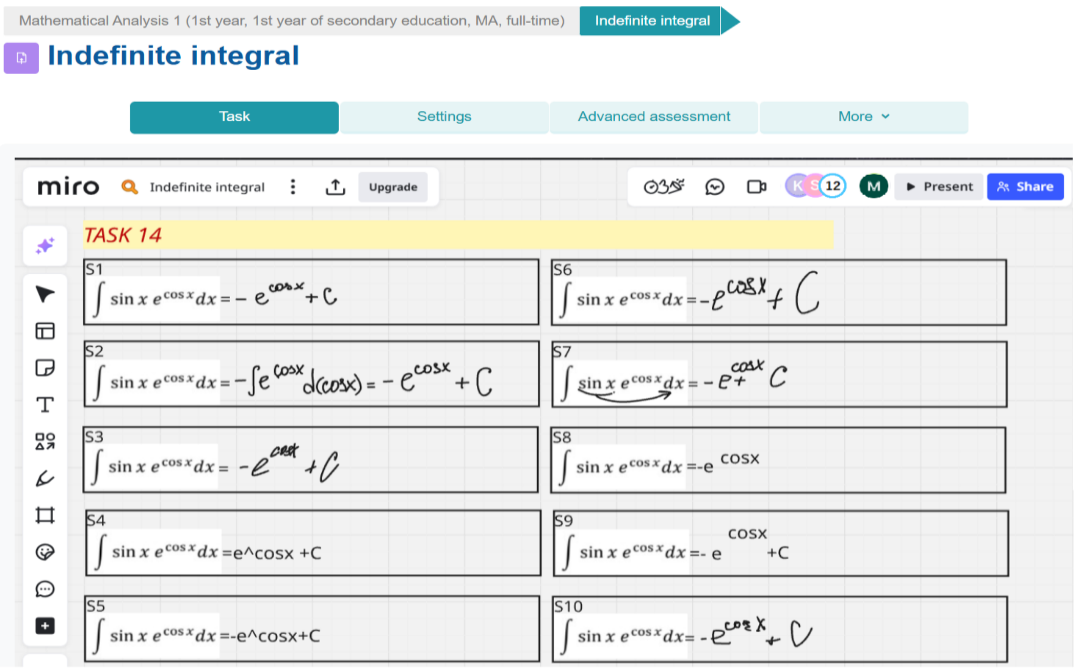

Figure 6. An example of student activity during a practical session using a Moodle Assignment with an embedded Miro Online Whiteboard

4.5 The results of the experiment

Next, we will proceed to answer the second research question,

> "What impact did the integrated conceptual and procedural approach to university-level mathematics education have on the results of students' mathematical training?"

4.5.1 Comparison of groups based on NMT results at the beginning of the experiment

Based on the comparison, it was found that U1 and U2 have similar levels of preparation in school mathematics. The average score for U2 students was 148.7, and for U1 students, it was 142.6, within the score range of 100 to 200, around 74.4% and 71.3%, respectively. According to the Mann-Whitney U test, it can be concluded that the groups did not significantly differ in their performance on the NMT in mathematics because U_{emp}=0, U_{kr}=11, and $U_{emp} < U_{kr}$ at a statistically significant level of $p \leq .01$. Here, U_{emp} represents the empirical value of the criterion, and U_{kr} is the critical value of the criterion.

4.5.2 Results of the diagnostic test

The diagnostic testing results (see Appendix 6) indicated that groups U1 and U2 possess similar levels of mathematical knowledge, as depicted in the histogram (refer to Fig. 7).

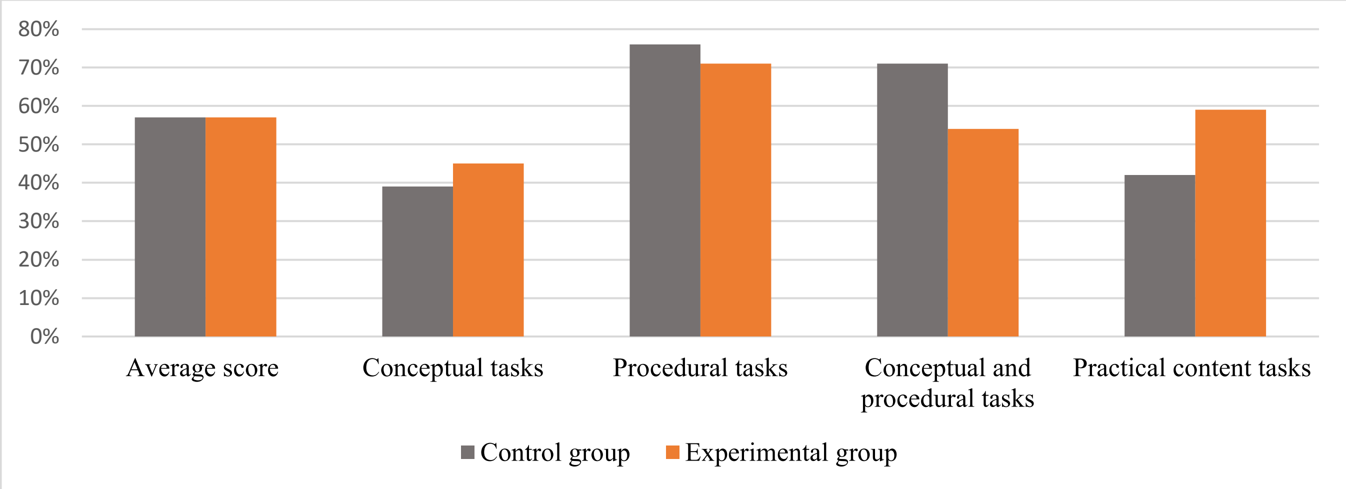

Figure 7. Results of the diagnostic test

In both groups, the average percentage of points actually scored out of a maximum of 40 is 57%. However, this average is significantly lower than the corresponding indicator according to the results of the NMT. The explanation is simple. NMTs are standardised tests that measure performance, focusing mainly on the ability to apply procedural skills to solve typical routine problems rather than testing an understanding of fundamental mathematical concepts or how particular procedures relate to the conceptual understanding of mathematics. Successful completion of the NMT test indicates that those who entered the first course have memorised the procedures needed to pass the test. However, this knowledge quickly degraded, as evidenced by the results of a diagnostic test at the university, which was administered only a few months after graduation but required a more conceptual understanding.

It has been confirmed through the analysis of student solutions that high school graduates face significant challenges in understanding mathematical concepts and applying them to real-world problems. The percentage of points scored for 'conceptual' tasks is only 39% and 45% in U1 and U2, respectively. Similarly, the indicators for solving applied content problems are low, at 42% and 59% for U1 and U2. On the other hand, there are higher scores for the ability to solve 'procedural' tasks, with 76% in U1 and 71% in U2.

Without delving into a detailed analysis of all the mistakes made, it's important to note that they are typically linked to a misunderstanding of certain concepts (such as power with a fractional exponent), cause-and-effect relationships between specific facts, violations of mathematical logic in reasoning, a lack of understanding of the essence of mathematical proof, failure to translate contextual problems into mathematical language, and a reluctance to use visualisations as a tool for finding solutions and for argumentation and interpretation. We will present statistics and focus on the errors made when solving task 10 to provide a concrete example:

"Indicate the largest value of a at which the equation $\sin\left(x+\frac{\pi}{3}\right)=2a^2+5a-6$ *has roots."*

4 students, with 2 in each group, were unable to solve the problem. Only 2 students solved the problem correctly. 4 students did not understand the content of the double inequality, 6 students didn't realise what a parameter means. Additionally, 3 students had difficulty solving quadratic inequalities. After finding the roots of the corresponding quadratic equation, they either did not know what to do next or misused the information. Interestingly, only 3 students attempted to use graphical interpretation at different stages of the solution.

In Figures 8 and 9, we observe two students' solutions to the task, which contained several mistakes, especially procedural ones, due to a lack of understanding of the relevant mathematical concepts. This demonstrates a formal understanding of problems involving parameters, the nature of double inequalities, and the process for solving quadratic inequalities. This has led to mistakes in its use and a failure to use graphical interpretation.

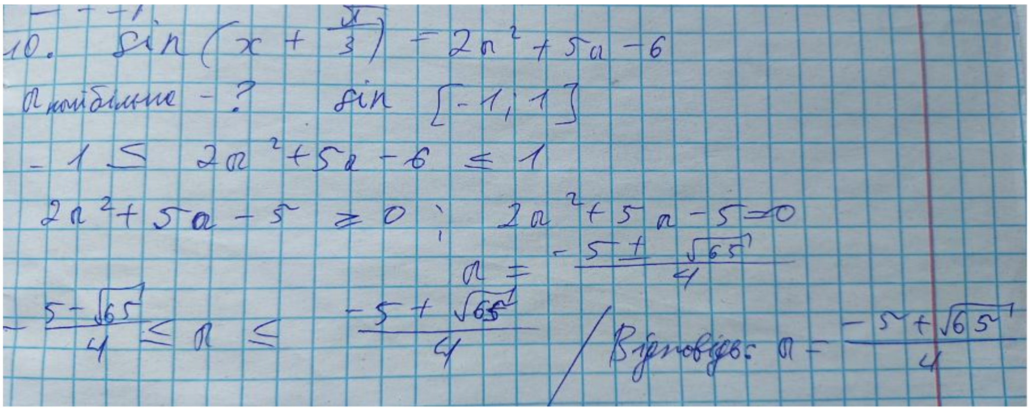

Figure 8. Solving the problem by the student

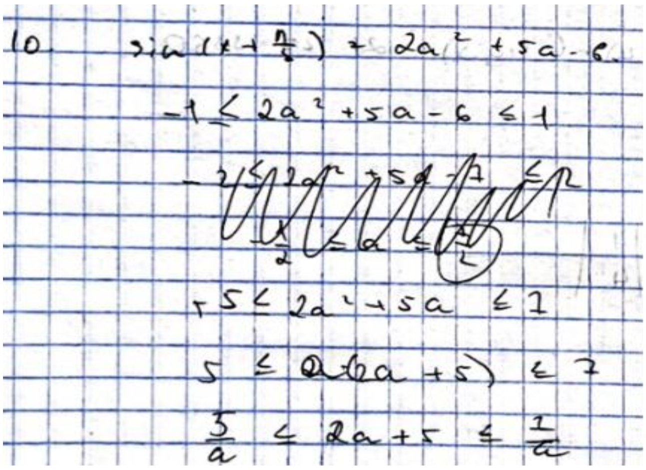

Figure 9. Solving the problem by the student

4.5.3 The exam results

Groups U1 and U2 in June 2024 took the exam on mathematical analysis. The histogram (Fig. 10) shows the exam results (A, B, C, D, E – grading scale; there were no F and FX grades).

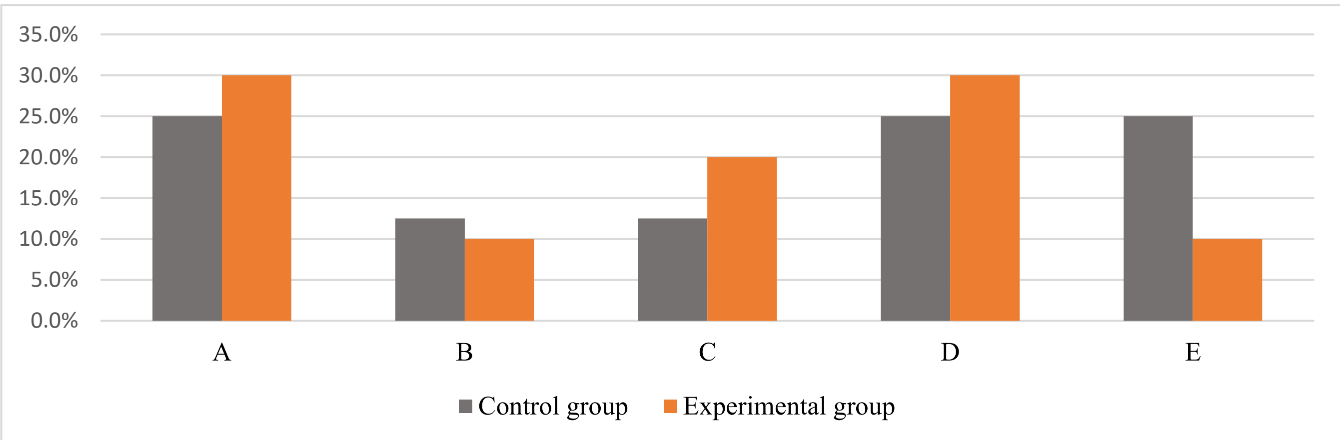

Figure 10. The exam results

The lecturers for this subject created exam questions for each group. Even though the results (number of grades (N) / percentages) for the mathematics analysis exam for U1 and U2 are slightly different (see Appendix 7), the tasks on the exam differ significantly in terms of the level of cognitive requirements, particularly the balance between conceptual and procedural knowledge required to complete the tasks.

For comparison, let's consider an example of a question from an exam about the geometric meaning of derivatives (a tangent line to the graph of a function).

U1: Determine the equation of the tangent line to the curve $f(x) = x^2 - 2x + 3$ at the point where x = 2.

U2: Prove that the segment of the tangent to the curve $y = \frac{k}{x}$, which is contained between the coordinate axes, is bisected by the point of contact.

It is evident that the task of the control group is purely procedural. It only requires formal knowledge of the tangent equation and the derivative calculation algorithm. This level of understanding is insufficient for solving the task of the experimental group. In the experimental group, proving the statement is necessary. For this, the student needed to conduct a study to make an important conclusion about the point of contact.

4.5.4 The results of the final test

The results of the final testing are given in Appendix 8.

In the table, the data analysis indicates that the average score for the set of conceptual understanding tasks is 1.23, which is 31.3% of the maximum possible 4 points for U1 and 2.64 points, or 66.0% for U2. For the set of procedurally fluency tasks, the relevant indicators are 2.30 points (55.8%) for U1 and 2.80 points (70.0%) for U2. Lastly, for the set of problem-solving tasks (modelling problems), the scores are 1.20 points (30.0%) for U1 and 2.44 points (61.0%) for U2 (refer to Fig. 11).

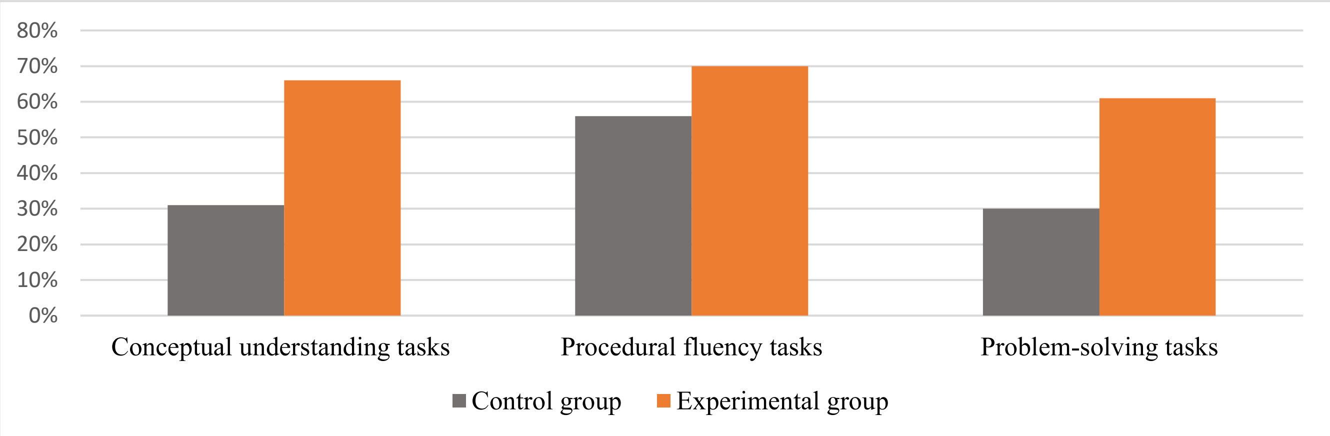

Figure 11. The results of the final test

The benefits of an integrated conceptual and procedural approach to learning, as opposed to a traditional approach, are demonstrated by the fact that U2 showed significantly higher indicators of conceptual understanding of mathematics and the ability to apply mathematics to solve real-world problems compared to U1. It's important to note that both groups started with similar conditions. Additionally, test results indicate that both groups continue to struggle with solving real-world problems.

5. DISCUSSION

In our study, we aimed to address two research questions.

(i) Why is it essential to integrate conceptual and procedural approaches in university-level mathematics education?

In our research, we did not refuse the importance of both procedural (e.g. Selden & Selden, 2011; Star, 2005, etc.) and conceptual approaches (e.g. Crooks & Alibali, 2014; Kholid et al., 2021; Niemi, 1996, etc.) to learning mathematics. However, we believe an integrated conceptual and procedural approach differs significantly from purely conceptual and purely procedural approaches. This approach combines theoretical knowledge with practical skills and emphasises mathematical applications in solving real problems, increasing students' motivation to study and their ability to use mathematics. Unlike conceptually oriented education, the integrated approach focuses on balancing theoretical knowledge and practical skills while necessitating an understanding of the concepts behind procedures. This allows for the conscious performance of procedures, the application of knowledge in new contexts, the adaptation of procedural skills to different situations, and the use of various approaches to problem-solving, as opposed to following specific algorithms. This approach encourages flexible thinking, fosters the development of individual strategies, and increases motivation to study.

Our research addresses the debate among scientists regarding the balance between conceptual and procedural knowledge in mathematics education (Stovner and Klette, 2022; Gándara & Contreras, 2009; Bossé & Bahr, 2008;) and also develops ideas for integrating procedural and conceptual knowledge in mathematics education (Ho, 2020; Phuong, 2020; Phuong, 2019; Rittle-Johnson et al., 2015; Kieran, 2013; Zahner et al., 2012; Wu, 1999; Hiebert, 1986). Our study responds to the work by Gerasimova et al. (2023). They highlighted the need for further research on combining conceptual and procedural teaching objectives and identifying the most effective teaching practices for strengthening the connection between these objectives and mathematics achievement. Emphasising the importance of the conceptual and procedural approach in learning mathematics, we assert that the sequence in which procedures and concepts are taught is crucial. This thesis is founded on our conviction that conceptual understanding forms the basis for productive thinking, allowing for selecting appropriate procedures for solving new problems, predicting solutions, gaining new insights, and developing new solution strategies. Simultaneously, performing procedures efficiently, confidently, and accurately enables effective research, focused argumentation, and often identifying patterns that spark new research ideas.

The research has led to the meaningful development of pedagogical practices, enabling an integrated conceptual and procedural approach based on the Inquiry-based learning (IBL) educational strategy.

(ii) What effect did the integrated conceptual and procedural approach to university-level mathematics education using Moodle have on students' mathematical training outcomes?

Our research confirmed the positive impact of an integrated conceptual and procedural approach in teaching and learning mathematics, implemented using the IBL strategy, on the students' mathematical training. This impact was evident by the results of final tests for both the control and experimental groups regarding absolute indicators of levels of conceptual understanding of mathematics, procedural knowledge, and the ability to use mathematics to solve problems, as well as the dynamics of these indicators.

We also interviewed students from the U2 group about their progress in learning mathematics and their understanding of the nature of mathematics. Here are some excerpts from their responses:

"When we began studying derivatives, it became apparent that I (and others!) had no idea what a derivative was, despite having studied it in school. Now, I not only understand how to calculate derivatives but also the origins of the formulas. I can also use derivatives to solve optimisation problems from various fields."

"I didn't really understand what a definite integral was in school. I only knew how to use the Newton-Leibnitz formula to calculate it. In class, we ended up figuring out the essence of the integral ourselves by calculating the area of a figure. Now I understand the connection between indefinite and definite integrals. This way of learning is much easier and very enjoyable and exciting."

"Previously, I saw mathematics as a collection of rules and formulas to be memorised. Now I realise that the most essential aspect of mathematics is understanding. I now view mathematics as a problem-solving tool and a method of action, not just for mathematical problems."

The students' feedback indicates a positive evaluation of the integrated approach to learning. We also inquired with the students about the reasons behind this remarkable progress. Here are the responses we received:

"I found understanding the essence of formulas and algorithms helpful before memorising them. I realised that memorising without understanding, which I often did in school, doesn't help me retain the information for long."

"Before approaching a new topic, we always made sure to understand what we were going to study and why. This approach was highly motivating and directed towards practical application."

"I find working in small groups instrumental. It allowed discussing the problem, listening to students' arguments, justifying and defending one's point of view, and stimulated to look for convincing arguments for this."

"Solving more complex problems than usual at school helped me to develop my mindset. It wasn't just about applying formulas; I had to think carefully, analyse, and even conduct studies."

Based on the students' responses, they noted all four applied practices when implementing an integrated conceptual and procedural approach.

In addition, students noted that an essential component of their progress was the use of Moodle digital tools, which made learning more structured, convenient, and accessible. According to them, having constant access to educational materials, tasks, and examples allowed them to organise their learning activities better, revisit challenging points, and gradually master new topics at a comfortable pace. The feedback function was especially valuable: comments on solutions, participation in peer assessment in the Workshop module, and discussions in forums helped them better understand their own mistakes, formulate arguments, and develop analytical thinking. Some students emphasised that even if they missed classes, they could catch up on the material thanks to access to the full course materials in Moodle, which contributed to the continuity and stability of their learning.

The results of our study not only confirmed the results of scientific investigations on the use of Moodle to form students' conceptual understanding or procedural knowledge in mathematics (Heba et al., 2014; Mlotshwa et al., 2020; Pahrudin et al., 2021; Psycharis et al., 2013 etc.) but also developed them in the context of implementing an integrated conceptual and procedural approach to learning mathematics.

The results of the final testing showed that the most significant difficulties in both groups arose when solving applied problems. The probable reason was the lack of experience in mathematical modelling, as such problems rarely occurred in school, and one year of study was insufficient to develop stable skills. The transition from the real context to the mathematical language posed the most significant difficulties, a finding also confirmed by other studies (Schaap et al., 2011).

The limitations of this study include its small sample size, as it was a pilot study intended to lay the groundwork for more extensive future research. Additionally, there was a limited selection of academic courses and specialities, and only one lecturer was involved in each group. The duration of the experiment and observation was also brief. Furthermore, we did not consider the influence of internal and external factors, particularly student motivation. As a result, we do not have sufficient grounds to claim that an integrated conceptual and procedural approach significantly enhances the level of mathematical preparation (for example, by one and a half to two times). Such a conclusion would require research on a larger scale.

However, we can confidently assert that the integration of conceptual understanding and procedural knowledge has a positive impact on student success, increases motivation, encourages conscious learning, and fosters the development of metacognition. This supports the effectiveness of the proposed approach, which is implemented using Moodle and can be applied to any mathematics course.

6. CONCLUSION

The study revealed that the conceptual and procedural approach in university-level mathematics education aligns with the understanding of mathematical competence, which involves a conceptual understanding of mathematical concepts, facts, methods, procedural fluency, and, based on these, the ability to solve problems. This conceptual and procedural triad serves as a source of support and a solid foundation for mathematical competence, enabling the practical application of mathematics to solve a wide range of mathematical and non-mathematical problems.

The implementation of an integrated conceptual and procedural approach, utilising specific pedagogical practices within Moodle, is substantiated and meaningfully described in the study. An e-learning course on mathematical analysis was designed in Moodle, which includes: clearly formulated learning outcomes; interactive lecture materials; training tasks for mastering standard procedures; mini-research and open-ended tasks to foster flexible thinking; forums for discussing solutions, errors and solution strategies; individual tasks with elements of self- and peer-assessment; self-testing modules with automatic feedback; tools for testing and assessing knowledge and skills.

The results of the study confirm that Moodle can be not just a technical tool, but also a powerful pedagogical platform that significantly enhances the implementation of a conceptual and procedural approach in the context of inquiry-based learning (IBL). Its capabilities contribute to a deeper understanding of mathematical concepts, the effective development of procedural skills, and the ability to think analytically and non-standardly when solving problems.

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BIOGRAPHICAL NOTES

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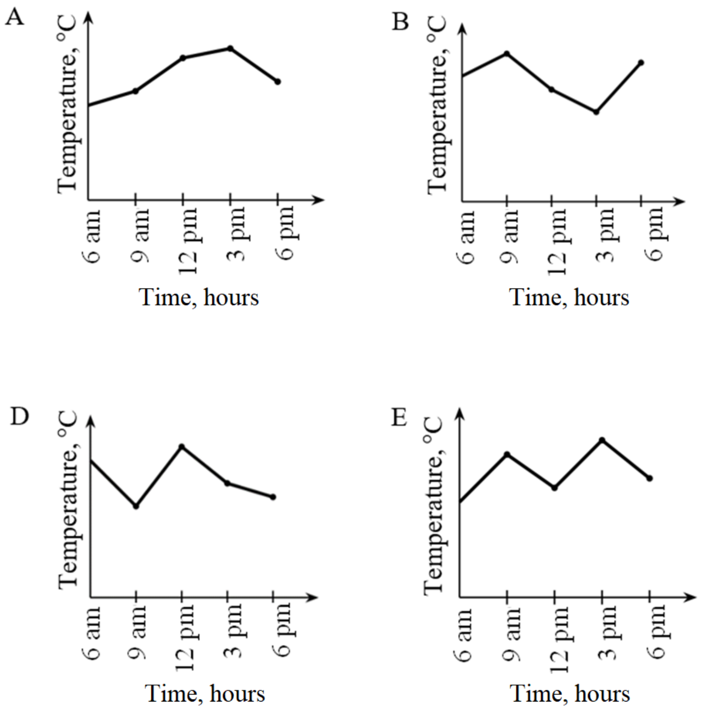
APPENDIX 1

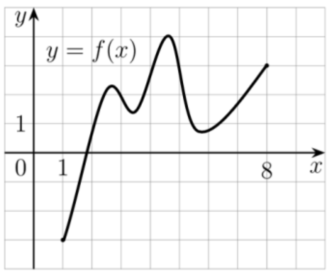
Diagnostic test
Complete the task with appropriate explanations. (Performance time – 90 min)

1. The table shows data on air temperature at different times of the same day. There is no air temperature scale (gradation) on the graphs. Which graph correctly displays the data given in the table?

Time, hours	6 a.m.	9 a.m	12 p.m.	3 p.m.	6 p.m.
Temperature, 0C	12	17	14	18	15



2. The figure shows the graph of the function f(x) defined on the interval [1, 8].



(a) How many zeros does the function have on the given interval?
(b) How many zeros does the derivative of this function have on the given interval?

3. Is the equality correct?

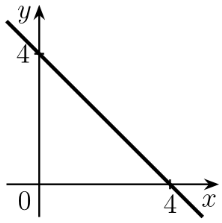
(a) $(-8)^{1/3} = -2$
(b) $\sqrt[3]{-8} = -2$

4. In the city centre, there are plans to install identical concrete flower vases in the form of rectangular parallelepipeds. These vases will have a height of 40 cm and dimensions of 50 cm in length and 40 cm in width. The side walls will have a thickness of 5 cm, and the bottom will have a thickness of 10 cm. How many cubic meters of concrete are needed to produce 10 such vases? Approximate the result to the nearest hundredth.
5. Prove that the sum of two even numbers is an even number.

6. The winner of the Maths Olympiad is gifted a set of 5 books, which includes 2 collections of Olympiad problems and 3

popular science books. If there are 8 different Olympiad problem books and 10 different popular science books, how many options exist for forming such a set of books?

7. Specify the equation of the line corresponding to the graph shown in the figure.



8. The denominator of a geometric progression is ⅔, and the sum of the first four terms is 65. Find the first term of this progression.

9. The airline's baggage regulations are as follows:
   - Economy class: 1 piece of baggage weighing up to 23 kg.
   - Business class: 2 pieces of baggage weighing up to 32 kg each.
   - Premium class: 3 pieces of baggage weighing up to 30 kg each.

   The prices are:
   - Economy class: UAH 5845
   - Business class: UAH 6370
   - Premium class: UAH 22906.

Which option is more cost-effective: buying two business-class or two economy-class tickets? Please note that the fee for excess baggage is 85 UAH per kilogram. You are carrying two suitcases, weighing 31 kg and 35 kg.

10. Determine the largest value of a for which the equation has roots
    $\sin\left(x+\frac{\pi}{3}\right) = 2a^2 + 5a - 6.$

* The students were not provided with information about categorising tasks into groups. They received tasks in a continuous list, where tasks 2, 3, and 10 were conceptual, 1, 6, and 8 were procedural, 5 and 7 were conceptual and procedural, and 4 and 9 were related to mathematical modelling problems.

APPENDIX 2

Task type	Level 0	Level 1	Level 2	Level 3
Procedural	There is no procedure required to complete the task	One standard procedure is sufficient to complete the task	Two different types of procedures are needed to complete the task	Completing the task requires the complex use of multiple interconnected procedures
Conceptual	Formally, the solution can be found using standard procedures (such as a formula) without delving into the essence of the pertinent mathematical concepts	The solution requires understanding at least one mathematical concept or fact, and its incorrect understanding can also lead to procedural mistakes	The solution requires understanding specific mathematical concepts on which a conscious and reasoned selection of appropriate procedures is based	The solution requires justifications, building a chain of arguments, and possibly considering different cases

Table 2A. Level scheme for determining the type of mathematical task

APPENDIX 3

	Task 1	**Task 2**	**Task 3**	**Task 5**	**Task 6**	**Task 7**	**Task 8**	**Task 10**
A	P2 K1	P0 K3	P0 K1	P2 K2	P3 K1	P2 K2	P2 K0	P2 K3
E_1	P3 K0	P0 K2	P0 K1	P2 K2	P3 K1	P2 K2	P2 K0	P2 K3
E_2	P2 K1	P0 K3	P0 K2	P2 K2	P3 K1	P2 K2	P1 K0	P3 K3

Table 3A. Results of assessment of tasks by authors and experts

* The levels of 'procedurality' (Pi) and 'conceptuality' (Kj) are assigned to tasks (i,j= 0,1,2,3). E_1 and E_2 are experts, and A is an author. According to the evaluations in the table, tasks 2, 3, and 10 are conceptual, tasks 1, 6, and 8 are procedural, and tasks 5 and 7 are conceptual and procedural.

APPENDIX 4

Test Task Evaluation Criteria

4 points: A correct answer with necessary justification and a detailed explanation of the solution process.
3 points: Correct answer and solution process without detailed explanation.
2 points: Partial completion of the solution or correct answer without necessary reasoning.
1 point: An incorrect solution process, which has led to an incorrect answer.

No points will be awarded if a student has not attempted the task.

APPENDIX 5

Final test

The assessment for evaluating mathematical skills in mathematical analysis course comprises 15 tasks - 5 for each section. Each task requires a reasoned and explained answer, with complete solutions provided. Whenever possible, utilise various solution methods (or justify your chosen method) and modes of presentation (such as tables, diagrams, graphs, symbolic notations, etc.). The test is to be completed in written form without using aids or tools, including digital devices.

Section I

1.1 Are 'equal sets' and 'equivalent sets' identical?

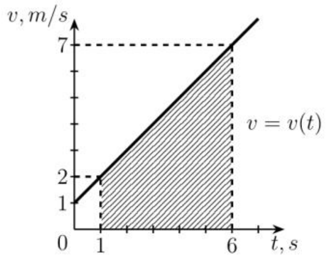
1.2. Can a continuous function *f*(*x*) on the interval [-10; 10]:
a) obtain the value of 2024
b) obtain any value A Є R
c) have R as the function's domain?

1.3. In the 'solution' provided for the following problem:
"Write the equation of the tangent to the curve $f(x) = \sqrt[3]{x^2}$ at the point with abscissa $x_0 = -8$", there is a mistake. Find it.

The solution.
The equation of the tangent to the curve f(x) at the point with abscissa x_0 has the form
$y = f(x_0) + f'(x_0)(x - x_0)$.
In our case, $f'(x) = \left(\sqrt[3]{x^2}\right)' = \left(x^{\frac{2}{3}}\right)' = \frac{2}{3}x^{-\frac{1}{3}}$. Since the expression $(-8)^{-\frac{1}{3}}$ is undefined, $f'(-8)$ does not exist, and the tangent to the curve $f(x) = \sqrt[3]{x^2}$ at the point with x-coordinate $x_0 = -8$ does not exist.

1.4 Sketch a graph of the function $f(x)$ that is continuous on R, with $f(0) = -2$, $f'(0) = 0$, $f(2) = 0$, $f'(2) > 0$, and $f''(2) = 0$.

1.5 The Figure depicts the graph of velocity v of a body moving in a straight line. Use the graph to determine the body's path from t = 1 s to t = 6 s.



Section II

2.1 Prove the equality of the sets (AUB)\C=(A\C)U(B\C).

2.2 Evaluate $f'(0)$, if $f(x) = \left(\frac{1+\sin x}{\cos x}\right)^2$.

2.3 How many roots does the equation have: $\sqrt{x+a} = -x$?

2.4 Find the area enclosed by the curve $y = -x^2 + 2x + 3$, its tangent at $x = 2$, and the x-axis.

2.5 Prove that the sum of odd functions on an interval is an odd function on that interval. Also, clarify whether the sum of even and odd functions can be odd.

Section III

3.1 Out of 40 programmers, 18 know the Python language, 19 know C++, and 21 know Java. It is known that ten programmers know both Python and C++, 7 know Python and Java, and 8 know C++ and Java. Three programmers do not know any of the languages. Please find the number of programmers who know all three programming languages simultaneously.

3.2 Find the right triangle of the largest area, given the sum of its legs is constant.

3.3 The rate of spread of the flu epidemic is described by the formula $N(t) = 0.003t(50 - t)$, where t represents time in days, and $N(t)$ represents the number of people who fell ill on the t-th day. On what day will the epidemic reach its maximum? How many days will it take for the epidemic to die out?

3.4 There is a straight road running alongside the forest. Picture yourself in the forest, 5 km away from the road and 13 km from your house, which stands by the road. You can travel through the forest at a speed of 3 km/h or along the road at 5 km/h. Your task is to move straight to the road and follow it to get home. How long will it take you at least to go home?

3.5 The concentration of the medicine in the body decreases over time due to natural elimination. The rate of decrease is given by $v(t) = -100e^{-t}$, where t represents time in hours. Find the equation for the change in the amount of the medicine in the patient's blood, given that the initial amount is 200 corresponding units.

APPENDIX 6

Tasks / Students	1	2	3	4	5	6	7	8	9	10	%
S1	4	2	1	1	1	4	3	4	4	4	70.0
S2	4	3	1	2	4	4	4	4	2	3	77.5
S3	3	2	1	1	1	4	3	4	4	3	65.0
S4	2	0	1	1	2	1	2	3	2	1	37.5
S5	4	3	1	3	4	3	4	4	1	3	75.0
S6	2	1	1	1	3	3	3	4	0	0	45.0
S7	1	1	1	0	4	0	0	0	3	0	25.0
S8	4	2	1	1	4	3	3	4	1	1	60.0
Average	**3.00**	**1.75**	**1.00**	**1.25**	**2.88**	**2.75**	**2.75**	**3.38**	**2.13**	**1.88**	**56.9**

Table 6A. Results of diagnostic testing, U1 (control group)

The average score for all tasks is 2.28 (or 57% of the maximum possible score of 4 points).

For conceptual tasks, the average score is 1.54 (39%); for procedural tasks, 3.04 (76%); for conceptual and procedural tasks, 2.82 (71%); for tasks with practical content (Nos. 4, 9), the average score is 1.69 (42%).

Tasks / Students	1	2	3	4	5	6	7	8	9	10	%
S1	3	2	1	2	2	3	4	4	4	2	67.5
S2	4	3	2	4	4	4	3	4	2	3	82.5
S3	3	3	1	2	1	3	2	4	4	3	65.0
S4	2	1	1	1	2	1	1	3	2	2	40.0
S5	4	3	2	3	3	2	4	4	1	3	72.5
S6	4	2	1	3	1	3	3	4	2	1	60.0
S7	1	1	1	1	2	2	2	1	3	0	35.0
S8	4	2	2	2	3	4	3	4	4	1	72.5
S9	3	3	1	1	1	2	1	3	2	2	47.5
S10	2	3	2	3	1	1	1	2	3	2	50.0
S11	1	2	1	1	1	2	2	2	2	0	35.0
Average	**2.82**	**2.27**	**1.36**	**2.09**	**1.91**	**2.45**	**2.36**	**3.18**	**2.64**	**1.73**	**57.0**

Table 6B. Results of diagnostic testing, U2 (experimental group)

The average score for all tasks is 2.28 (or 57% of the maximum possible score of 4 points).

For conceptual tasks, the average score is 1.79 (45%); for procedural tasks, 8.82 (71%); for conceptual and procedural tasks, 2.14 (54%); for tasks with practical content (Nos. 4, 9), the average score is 2.37 (59%).

APPENDIX 7

Grades / Groups	A N / %	B N / %	C N / %	D N / %	E N / %	FX N / %	F N / %
U1	2 / 25	1 / 12.5	1 / 12.5	2 / 25	2 / 25	0 / 0	0 / 0
U2	3 / 30	1 / 10	2 / 20	3/ 30	1 / 10	0 / 0	0 / 0

Table 7A. Results of the mathematical analysis exam

APPENDIX 8

	1. Conceptual understanding						2. Procedural fluency						3. Problem-solving (modelling)					
T	**1.1**	**1.2**	**1.3**	**1.4**	**1.5**	**%**	**2.1**	**2.2**	**2.3**	**2.4**	**2.5**	**%**	**3.1**	**3.2**	**3.3**	**3.4**	**3.5**	**%**
St																		
S1	1	2	0	0	1	20	2	4	1	3	2	60	3	0	2	1	0	30
S2	4	2	1	2	4	65	3	4	1	4	3	75	4	2	4	1	1	60
S3	0	1	1	2	1	25	2	3	3	2	2	60	2	1	2	1	0	30
S4	1	1	0	1	1	20	1	1	1	2	1	30	1	1	2	0	0	20
S5	3	1	2	1	2	45	3	4	2	3	3	75	2	2	3	1	0	40
S6	1	1	0	1	1	20	2	4	1	3	1	55	1	2	1	0	0	20
S7	3	1	1	1	1	35	2	2	1	1	1	35	0	0	1	1	0	10
S8	1	1	0	1	1	20	2	4	1	3	1	55	1	1	3	1	0	30
Av	1.75	1.25	0.63	1.13	1.38	31.3	2.13	3.25	1.38	2.63	1.75	55.6	1.75	1.13	2.25	0.75	0.13	30,0

Table 8A. Final test results (control group)

T= tasks, St=students, Av=average

The average score on tasks for conceptual understanding: 1.23 (31% of the maximum value of 4 points); for tasks on procedural fluency: 2.30 (56%); for mathematical modelling tasks: 1.20 (30%).

	1. Conceptual understanding						2. Procedural fluency						3. Problem-solving (modelling)					
T	**1.1**	**1.2**	**1.3**	**1.4**	**1.5**	**%**	**2.1**	**2.2**	**2.3**	**2.4**	**2.5**	**%**	**3.1**	**3.2**	**3.3**	**3.4**	**3.5**	**%**
St																		
S1	4	2	1	2	4	65	3	4	3	3	2	75	4	2	4	1	2	65
S2	4	2	2	3	4	75	4	4	3	4	3	90	4	3	4	2	4	85
S3	4	2	3	1	3	65	2	4	2	3	2	65	1	2	3	2	2	50
S4	3	1	1	3	4	60	3	4	2	3	2	70	2	2	3	2	3	60
S5	4	2	3	2	4	75	4	4	3	3	2	80	4	4	3	2	2	75
S6	4	2	1	3	3	65	2	3	2	3	1	55	2	1	3	2	2	50
S7	2	2	0	2	4	50	2	3	1	3	1	50	3	2	2	1	1	45
S8	4	2	1	2	4	65	2	3	1	4	2	60	2	1	3	2	1	45
S9	4	3	4	3	4	90	4	4	4	4	3	95	4	3	4	2	4	85
S10	2	1	1	2	4	50	3	4	1	2	2	60	1	2	3	2	2	50
Av	3.5	1.9	1.7	2.3	3.8	66,0	2.9	3.7	2.2	3.2	2.0	70.0	2.7	2.2	3.2	1.8	2.3	61.0

Table 8B. Final test results (experimental group*)

The average score on tasks for conceptual understanding: 2.64 (66% of the maximum value of 4 points); for tasks on procedural fluency: 2.80 (70%); for mathematical modelling tasks: 2.44 (61%).

*10 students were tested (1 student in the first semester was excluded due to departure to another country)