# Optimal Cybersecurity Management: Global Controllability of Linear Models*

Mariia Astafieva[1,*,†] and Nataliya Stepanenko[2,†]

[1] Borys Grinchenko Kyiv Metropolitan University, 18/2 Bulvarno-Kudriavska str., 04053 Kyiv, Ukraine
[2] National Technical University of Ukraine "Igor Sikorsky Kyiv Polytechnic Institute," 37 Peremohy ave., 03056 Kyiv, Ukraine

**Abstract**
In the age of digitalization, cybersecurity is critical for modern technological systems. The tasks of protecting information nodes, preventing the spread of cyberattacks, and restoring system functionality after successful attacks require studying the possibility of controlling dynamic processes. Mathematical models based on ordinary differential equations make it possible to describe and analyze these processes in terms of controllability. An extremely important property of cybersecurity models is their global controllability, which means that the system can be moved from any initial state to a desired end state using appropriately selected control. In the context of cybersecurity, this allows for an effective response to threats, recovery from attacks, and prevention of undesirable scenarios. This paper presents the conditions for global controllability of stationary and non-stationary linear systems of differential equations that model dynamic processes in the information space. The results obtained by different researchers are systematized, and the author presents his proof of some of them. Examples confirming the theory are constructed.

**Keywords**
cybersecurity, mathematical model, system of linear differential equations, mathematical control theory, global controllability of the model

## 1. Introduction

The mathematical theory of control is very important and relevant today. We always strive to have control over a process or physical system to make it behave optimally, minimize risks, eliminate threats, etc. The theory of optimal control is precisely concerned with analyzing and finding solutions for optimal control of a system or process [1].

One of the biggest threats to information and cyber security is malware. Dynamic processes of information dissemination are often described by systems of differential equations. A separate class of such models that are interesting in the context of our study are compartmental models based on ordinary differential equations. They typically describe the dynamics of malware propagation and have been studied in many papers, for example [2–4]. Studies of various cybersecurity models based on the Lotka-Volterra model can be found, for example, in [5–9], and differential models of information dissemination and information confrontation are considered in [10]. Thus, ordinary differential equations are important tools for analyzing and controlling dynamic systems, such as cyberattacks and defense mechanisms. A defense system must be controllable. It is always necessary to have information about what is happening in the information system, or even better, to get a forecast of the situation, predict the behavior and evolution of malware, and understand the effectiveness of various countermeasures. This is where the mathematical theory of control, described in many books, such as [11–14], comes in. Based on this theory, various optimal control problems are studied, including those in cybersecurity, as exemplified in [4, 15–17]. The proposed paper is devoted to the problem of global manageability, which means the ability to fully control

* CPITS 2025: Workshop on Cybersecurity Providing in Information and Telecommunication Systems, February 28, 2025, Kyiv, Ukraine
* Corresponding author.
† These authors contributed equally.
m.astafieva@kubg.edu.ua (M. Astafieva); nataliya.stepanenko@lll.kpi.ua (N. Stepanenko)
0000-0002-2198-4614 (M. Astafieva); 0000-0002-9690-4797 (N. Stepanenko)

the security system to: (a) eliminate threats; (b) ensure system stability; and (c) prevent the spread of attacks. The key reasons why global manageability is important are:

- Managing the spread of threats (it is important to be able to bring a system infected with, for example, a virus or other malware to a secure state; global manageability ensures that this is possible for any initial threat configuration).
- Adaptation to new attacks (cyber threats are constantly changing, so security systems must be able to adapt; global controllability allows you to adjust the protection parameters, in particular, the control input u(t), to take into account new types of attacks).
- Recovery after an attack (after a successful attack, it is necessary to have mechanisms that allow the system to return to the desired state; global manageability allows this to be done even in complex multi-component systems).
- Optimization of resources (in systems with limited resources, such as computing, financial, or human resources, global controllability allows to determine the minimum required set of control actions to achieve security goals).
- Building resilient systems (global controllability contributes to the development of resilient systems that can remain under control even in the event of significant disturbances or changes in the system, such as large-scale cyber-attacks).

Thus, global controllability in cybersecurity systems is a fundamental property that allows not only responding to attacks but also actively maintaining the system's stability in the face of ever-growing threats [18, 19].

**The purpose of the paper** is to consider the conditions of global controllability of a dynamic model of cybersecurity described by a linear system of ordinary differential equations.

## 2. Scalar equation with a vector control function

Let the control object be described by a linear differential equation

$$\dot{x}=a(t)x+b_1(t)u_1(t)+...+b_m(t)u_m(t) \tag{1}$$

where $\dot{x}=\dfrac{dx}{dt}$, $u(t)=(u_1(t),\dots,u_m(t))$ is the control vector function, defined and continuous on a segment [0,1], i.e., $u(t)\in C[0,\,1]$. This function stabilizes the system's functioning and counteracts cyber attacks.

**Definition 1.** Equation (1) is said to be *globally controllable* on the interval [0,1], if for any fixed values $x_0,\ x_1\in R$ there exists a vector function $u=u(t)\in C[0,\,1]$, such that equation (1) has a solution $x=x(t)$, that satisfies the boundary conditions $x(0)=x_0,\ x(1)=x_1$.

Let us find out the conditions of the global controllability of equation (1). It is necessary to choose a function $\sum_{j=1}^{m} b_j(t)u_j(t)=f(t)$ so that the conditions $x(0)=x_0,\ x(1)=x_1$ for any predetermined ones are fulfilled $x_0,\ x_1\in R$.

(1) is the heterogeneous equation of the form $\dot{x}=a(t)x+f(t)$. As is well known, for fixed functions $a(t), f(t)\in C[0,\ 1]$ with an initial condition $x(0)=x_0$ this equation has a unique solution

$$x=x(t)=e^{\int_0^t a(\sigma)d\sigma}\left(x_0+\int_0^t e^{-\int_0^\tau a(\sigma)d\sigma} f(\tau)d\tau\right).$$

Taking into account this, as well as the requirement that the solution of equation (1) should, in addition to the condition $x(0)=x_0$, satisfy the condition $x(1)=x_1$ for any predetermined point $x_1\in R$, we write:

$$x_1=x(1)=e^{\int_0^1 a(\sigma)d\sigma}\left(x_0+\int_0^1 e^{-\int_0^\tau a(\sigma)d\sigma}\langle b(\tau), u(\tau)\rangle d\tau\right) \tag{2}$$

where marked $\langle b(\tau),\ u(\tau)\rangle=\sum_{j=1}^{m} b_j(t)u_j(t)$. Let's write (2) in the following form:

$$\int_0^1 e^{-\int_0^\tau a(\sigma)d\sigma}\langle b(\tau), u(\tau)\rangle d\tau = y \tag{3}$$

where $y=x_1 e^{-\int_0^1 a(\sigma)d\sigma}-x_0$.

Since the values of $x_0$ and $x_1$ change $R$ arbitrarily and independently of each other, and takes on arbitrary values with $R$. Thus, the global controllability of equation (1) on the interval [0,1] is equivalent to the fact that the integral equation (3) has a solution $u=u(t)\in C[0,\ 1]$ for any value $y\in R$.

It is easy to verify the validity of the following statement.

**Theorem 1.** For the integral equation (3) to have a solution $u=u(t)\in C[0,\ 1]$ it is necessary and sufficient that the condition is fulfilled

$$G=\int_0^1 e^{-2\int_0^\tau a(\sigma)d\sigma}\sum_{j=1}^{m} b_j^2(\tau)d\tau \neq 0 \tag{4}$$

.

***Proof.***

Indeed, if the condition (4) is fulfilled, then, obviously, the integral equation (3) also has solutions for each fixed $y\in R$. One such solution looks like this

$$u=u_0(\tau)=b(\tau)\cdot e^{-\int_0^\tau a(\sigma)d\sigma}\cdot\frac{y}{G}.$$

Other solutions $u = u(\tau)$ can be presented in the form of the sum $u = u_0(\tau) + v(\tau)$, where $v(\tau)$ is a continuous vector function that is a solution of the integral equation

$$\int_0^1 e^{-\int_0^\tau a(\sigma)d\sigma} \langle b(\tau), v(\tau)\rangle d\tau = 0.$$

If condition (4) is not fulfilled, then this means that $b_j(\tau) = 0,\ j = \overline{1, m},$ for all $\tau \in [0,\ 1]$. At the same time, (2) takes the form $x_1 = x_0 e^{\int_0^1 a(\sigma)d\sigma}$ and, obviously, cannot be fulfilled for any $x_0,\ x_1 \in R$.

The theorem is proved.

## 3. A system of linear equations with a vector control function

Let the control object be described by a system of differential equations

$$\dot{x} = A(t)x + B(t)u(t) \qquad (5)$$

where $x \in R^n$, $u = u(t) \in R^m$ is the control function, *A(t)* is a square matrix of dimension $n \times n$, which indicates the degree of threat of information impact and whose elements are real scalar functions $a_{ij}(t)$ defined and continuous on the interval $(a,\ b)$ ($a$ and $b$ maybe infinite); the matrix $B(t)$, that sets the degree of system security is rectangular, consists of n rows and m columns, its elements are continuous on $(a,\ b)$ scalar functions. The elements of matrices are formed by cybersecurity experts.

**Definition 2.** System (5) will be called *globally controllable* on a segment $[t_0,\ t_1]$ $([t_0,\ t_1] \subset (a,\ b))$ if for any fixed values $x_0,\ x_1 \in R$ exists a vector function $u = u(t) \in C[t_0,\ t_1]$, in which the system has a solution $x = x(t)$, that satisfies the boundary conditions $x(t_0) = x_0,\ x(t_1) = x_1$.

Let's write down what the solution of system (5) looks like. It is a heterogeneous system. A homogeneous system corresponds to it

$$\dot{x} = A(t)x. \qquad (6)$$

Let us denote $\Omega_{t_0}^t$ the fundamental matrix of solutions of this system, normalized at the point $t = t_0$, $\Omega_{t_0}^t|_{t=t_0} = I_n$, $I_n$ is an *n*-dimensional unitary matrix. Knowing that the solutions of a heterogeneous system $\dot{x} = A(t)x + f(t)$, where $f(t)$ is some vector function, defined and continuous on the interval $(a,\ b)$, are given by equality

$$x = x(t) = \Omega_{t_0}^{t}\left( x_0 + \int_{t_0}^{t} \Omega_{\tau}^{t_0} f(\tau) d\tau \right) \tag{7}$$

where $x_0 \in R^n$ is an arbitrarily fixed constant vector and $x(t_0) = x_0$, we write down the solution of system (5) under the condition that $u(t)$ it is continuous:

$$x(t) = \Omega_{t_0}^{t}\left( x_0 + \int_{t_0}^{t} \Omega_{\tau}^{t_0} B(\tau) u(\tau) d\tau \right). \tag{8}$$

The following statement is true.

**Theorem 2.** For system (5) to be globally controllable on the interval $[t_0, t_1]$ it is necessary and sufficient that the condition

$$\det G[t_0, t_1] \neq 0 \tag{9}$$

where $G[t_0, t_1] = \int_{t_0}^{t_1} \Omega_{\tau}^{t_0} B(\tau) B^T(\tau) \left(\Omega_{\tau}^{t_0}\right)^T d\tau$ is the Gram matrix.

***Proof.***

*Sufficiency.* Suppose that the system (5) $u(t)$ is a continuous vector function. Then the solution of system (5) is of the form (8). We need the condition to be satisfied $x(t_1) = x_1$, which means that for any fixed constant vectors $x_0, x_1 \in R^n$, the system of integral equations must-have solutions

$$\Omega_{t_1}^{t_0} x_1 - x_0 = \int_{t_0}^{t_1} \Omega_{\tau}^{t_0} B(\tau) u(\tau) d\tau. \tag{10}$$

When the condition (9) is fulfilled, one of these solutions is

$$u(\tau) = B^T(\tau)\left(\Omega_{\tau}^{t_0}\right)^T \left(G[t_0, t_1]\right)^{-1}\left(\Omega_{t_1}^{t_0} x_1 - x_0\right). \tag{11}$$

Indeed, by substituting (11) into the right-hand side of equation (10), we will have:

$$\int_{t_0}^{t_1} \Omega_{\tau}^{t_0} B(\tau) B^T(\tau) \left(\Omega_{\tau}^{t_0}\right)^T \left(G[t_0, t_1]\right)^{-1} \left(\Omega_{t_1}^{t_0} x_1 - x_0\right) d\tau = \left(G[t_0, t_1]\right) \times \left(G[t_0, t_1]\right)^{-1} \times \left(\Omega_{t_1}^{t_0} x_1 - x_0\right)$$

$$= \Omega_{t_1}^{t_0} x_1 - x_0.$$

*Necessity.* Let the system (5) be globally controllable, but, at the same time, the condition (9) is not fulfilled, i.e., $\det G[t_0, t_1] = 0$. This means that there exists a nonzero constant vector such that

$$G[t_0, t_1] \cdot \eta = 0. \tag{12}$$

The vector $\eta$ can be chosen to be a unit. Then

$$\|\eta\| = \sqrt{\langle \eta, \eta \rangle} = 1. \tag{13}$$

Since we assumed that the system (5) is globally controlled on the interval $\left[t_0,\, t_1\right]$, then the system of integral equations (10) $\Omega_{t_1}^{t_0} x_1 - x_0 = \eta$ has a solution $u = \widetilde{u}(\tau)$, i.e., the equality holds

$$\int_{t_0}^{t_1} \Omega_{\tau}^{t_0} B(\tau) \widetilde{u}(\tau) d\tau = \eta \tag{14}$$

Note that, based on equality (12), we can assert that the value of the quadratic form $\langle G\left[t_0,\, t_1\right] x, x\rangle$ at $x = \eta$ is zero. So, we have:

$$0 = \langle G[t_0, t_1] x, x\rangle = \int_{t_0}^{t_1} \langle M^T(\tau) M(\tau)\eta, \eta\rangle d\tau = \int_{t_0}^{t_1} \|M(\tau)\eta\|^2 d\tau \tag{15}$$

where is marked $M(\tau) = B^T(\tau)\left(\Omega_{\tau}^{t_0}\right)^T$.

Therefore, the identity must hold

$$M(\tau)\eta \equiv 0 \ \forall \tau \in [t_0, t_1]. \tag{16}$$

It follows from (14) and the identity (16)

$$\|\eta\|^2 = \eta^T \cdot \eta = \left[\int_{t_0}^{t_1} \Omega_{\tau}^{t_0} B(\tau) \widetilde{u}(\tau) d\tau\right]^T \times \eta = \int_{t_0}^{t_1} \left(\widetilde{u}(\tau)\right)^T M(\tau) d\tau \cdot \eta = 0,$$

and this contradicts (13). Therefore, condition (9) follows from the global controllability of system (5). The necessity, and therefore the entire theorem, is proved.

The theorem and the process of its proof lead us to several conclusions.

1. From (15), we see that the Gram matrix is symmetric and non-negative, i.e., the inequality holds for all $\langle G\left[t_0,\, t_1\right] x, x\rangle \geq 0$. Moreover, condition (9) is equivalent to the following condition:

$$\langle G[t_0, t_1] x, x\rangle \geq \gamma \|x\|^2 \ \ \forall x \in R^n, \ \gamma = const > 0. \tag{17}$$

Indeed, for the symmetric Gram matrix $G\left[t_0,\, t_1\right]$ the condition is fulfilled $\langle G\left[t_0,\, t_1\right] x, x\rangle \geq 0$.

This means that all eigenvalues of the Gram matrix are real and nonnegative. If, in addition, condition (9) is fulfilled, then all eigenvalues are positive, and this means that condition (17) is fulfilled.

2. The global controllability of the system (5) on the interval $\left[t_0,\, t_1\right]$ is equivalent to the existence on this segment of a solution $u = u(\tau)$ of the system of integral equations $y = \int_{t_0}^{t_1} \Omega_{\tau}^{t_0} B(\tau) u(\tau) d\tau$ for each fixed $y \in R^n$.

3. If the system (5) is globally controllable on the interval $\left[t_0,\, t_1\right]$, then it is globally controllable on any interval $\left[\widetilde{t}_0,\, \widetilde{t}_1\right]$ such that $\left[t_0,\, t_1\right] \subset \left[\widetilde{t}_0,\, \widetilde{t}_1\right] \subset (a,\, b)$.

## 4. Controllability conditions of linear systems with smooth coefficients

It should be noted that finding the Gram matrix $G\left[t_0, t_1\right]=\int_{t_0}^{t_1} \Omega_\tau^{t_0} B(\tau) B^T(\tau)\left(\Omega_\tau^{t_0}\right)^T d\tau$ of system (5) is associated with certain difficulties, since it is difficult to write down the fundamental matrix $\Omega_{t_0}^t$ of the solutions of the corresponding homogeneous system is not always possible. It turns out that in the case of smooth matrices $A(t)$ and $B(t)$ (whose elements are continuously differentiable functions up to a certain order) there is a sufficient condition for the global controllability of the system (5), which does not require knowledge of the fundamental matrix of the system (6).

We assume that $A(t)\in C^{n-2}\left[t_0, t_1\right]$, $B(t)\in C^{n-1}\left[t_0, t_1\right]$, i.e., matrix elements $A(t)$ are continuously differentiable functions up to and including order $n-2$, and matrix elements are $B(t)$—up to $n-1$ and including order.

Let's enter the operator

$$\Delta\bullet = -A(t)\bullet + \frac{d\bullet}{dt}. \tag{18}$$

Let denote the matrix by $W(t)$

$$W(t)=\left(B(t), \Delta B(t), \Delta^2 B(t), \ldots, \Delta^{n-1} B(t)\right) \tag{19}$$

which consists of *n* rows and *nm* columns.

**Theorem 3.** Let there exist $\tilde{t}\in\left[t_0, t_1\right]$ such that the rank of the matrix $W(t)$ is equal to the number of its rows, i.e.,

$$rang W(\tilde{t})=n. \tag{20}$$

Then system (5) is globally controllable.

***Proof.*** First, consider the case when $A(t)\equiv 0$. In this case, the operator (18) is only a differentiation operator and the matrix (19) takes the form

$$W(t)=\left(B(t), \frac{d}{dt}B(t), \frac{d^2}{dt^2}B(t), \ldots, \frac{d^{n-1}}{dt^{n-1}}B(t)\right). \tag{21}$$

Since under the condition $A(t)\equiv 0$ fundamental solution matrix $\Omega_{t_0}^t ¿ I_n$, the Gram matrix has the form

$$G\left[t_0, t_1\right]=\int_{t_0}^{t_1} B(\tau)B^T(\tau)d\tau. \tag{22}$$

Assume that the matrix (22) is degenerate. Then there exists a nonzero vector $z \in R^n$ such that $G[t_0, t_1]z = 0$. It follows from this:

$$0 = \langle G[t_0, t_1]z, z\rangle = \int_{t_0}^{t_1} \langle B(\tau)B^T(\tau)z,\ z\rangle d\tau = \int_{t_0}^{t_1} \|B^T(\tau)z\|^2 d\tau,$$

and this is possible only in the case when $B^T(\tau)z \equiv 0\ \forall \tau \in [t_0, t_1]$. Thus, we have the identity $z^T B(\tau) \equiv 0$, by differentiating which we obtain the following identities:

$$z^T \frac{d}{dt} B(t) \equiv 0,\ z^T \frac{d^2}{dt^2} B(t) \equiv 0,\ \dots,\ z^T \frac{d^{n-1}}{dt^{n-1}} B(t) \equiv 0\ \ \forall t \in [t_0, t_1].$$

This implies a linear dependence of the rows of the matrix (21), which contradicts condition (20) for this matrix.

Now consider the general case when the matrix $A(t)$ is not identically equal to zero. Let's replace variables in the system (5)

$$x = \Omega_{t_0}^t y \tag{23}$$

where $\Omega_{t_0}^t$ is the fundamental solution matrix of the system (6). We will have

$$\dot{x} = \left(\frac{d}{dt}\Omega_{t_0}^t\right) y + \Omega_{t_0}^t \dot{y} = A(t)\Omega_{t_0}^t y + \Omega_{t_0}^t \dot{y} = A(t)\Omega_{t_0}^t y + B(t)u.$$

From here

$$\dot{y} = \widetilde{B}(t)u \tag{24}$$

where is indicated

$$\widetilde{B}(t) = \Omega_t^{t_0} B(t). \tag{25}$$

Thus, by replacing variables (23), system (5) is transformed into system (24), in which the first term is missing $\widetilde{A}(t)y$. Let us now find the matrix (21), which is replaced from $B(t)$ to $\widetilde{B}(t)$. To calculate the derivative of the inverse matrix, we differentiate the identity $\Omega_t^{t_0} \cdot \Omega_{t_0}^t \equiv I_n$. We get

$$\left(\frac{d}{dt}\Omega_t^{t_0}\right) \cdot \Omega_{t_0}^t + \Omega_t^{t_0} \times A(t)\Omega_{t_0}^t \times 0,$$

where

$$\frac{d}{dt}\Omega_t^{t_0} = -\Omega_t^{t_0} A(t). \tag{26}$$

Based on (26), we write down the derivative matrices (25):

$$\frac{d}{dt}\widetilde{B}(t) = \left(\frac{d}{dt}\Omega_t^{t_0}\right) B(t) + \Omega_t^{t_0}\left(\frac{d}{dt}B(t)\right) = -\Omega_t^{t_0} A(t)B(t) + \Omega_t^{t_0}\frac{d}{dt}B(t) = \Omega_t^{t_0} \Delta B(t),$$

$$\frac{d^2}{dt^2}\widetilde{B}(t) = \frac{d}{dt}\left(\Omega_t^{t_0}\Delta B(t)\right) = \Omega_t^{t_0}\Delta^2 B(t),\ \dots,\ \frac{d^{n-1}}{dt^{n-1}} \sim B(t) = \Omega_t^{t_0}\Delta^{n-1} B(t).$$

Thus, the matrix (21) in our case has the form

$$\bar{W}(t)=\Omega_t^{t_0}\left(B(t),\Delta B(t),\Delta^2 B(t),\ldots,\Delta^{n-1}B(t)\right).$$

Since the fundamental solution matrix is a nondegenerate matrix, the rank of the matrix $\bar{W}(t)$ is the same as the rank of the matrix $W(t)$. This completes the proof of the theorem.

## 5. Controllability conditions for linear systems with constant coefficients

Let's consider a system of differential equations with constant coefficients (which is most often the case in practice since the indicators of threat probabilities and system security are usually numerical)

$$\dot{x} = Ax + Bu, \tag{27}$$

where $A$ is a constant square matrix of dimension $n\times n$, $B$ is a constant rectangular matrix consisting of $n$ rows and $m$ columns, $u$ is a control vector function. The matrix (19) in this case is constant: $\left(B,-AB,A^2B,\ldots,(-1)^{n-1}A^{n-1}B\right)$. It is easy to see that changing the sign does not affect the rank of this matrix. Thus, based on Theorem 3, we can state that for the global controllability of the system (27) it is sufficient that

$$rang\left(B, AB, A^2B,\ldots, A^{n-1}B\right)=n. \tag{28}$$

This fact was established in the second half of the twentieth century by Kalman [20]. It turns out that equality (28) is not only sufficient but also a necessary condition for the global controllability of the system (27). That is, the following theorem holds.

**Theorem 4.** (*Kalman*) System (27) is globally controllable if and only if condition (28) is satisfied.

Let's prove the necessity.

The matrix $\Omega_{t_0}^t$ of a linear system $\dot{x}=Ax$ with constant coefficients, which corresponds to system (27), can always be written in the form:

$$\Omega_{t_0}^t = e^{A(t-t_0)} = I_n + A(t-t_0)+\frac{1}{2!}A^2(t-t_0)^2+\frac{1}{3!}A^3(t-t_0)^3+\ldots. \tag{29}$$

Let's assume that $rang\left(B,AB,A^2B,\ldots,A^{n-1}B\right)<n$. Then there exists a nonzero vector $\eta\in R^n$ such that $\eta^T W=0$, where $W=\left(B,AB,A^2B,\ldots,A^{n-1}B\right)$. This means that equalities are fulfilled

$$\eta^T B=0, \eta^T AB=0, \eta^T A^2B=0, \ldots, \eta^T A^{n-1}B=0. \tag{30}$$

We will show that then for any natural value $j$ the equality holds

$$\eta^T A^j B=0,\ j=1,2,\ldots. \tag{31}$$

We use the well-known Cayley-Hamilton theorem [21], which states that any square matrix satisfies its characteristic equation. That is, if the characteristic equation of the matrix $A$ $\det\left(A-\lambda I_n\right)=0$ is written in the form $\lambda^n+\alpha_1\lambda^{n-1}+\alpha_2\lambda^{n-2}+\ldots+\alpha_{n-1}\lambda+\alpha_n=0$, then the equality is correct

$$A^n+\alpha_1 A^{n-1}+\alpha_2 A^{n-2}+\ldots+\alpha_{n-1}A+\alpha_n I_n=0 \tag{32}$$

where 0 on the right-hand side means the zero matrix.

From equality (32) we have:

$$A^n B=-\alpha_1 A^{n-1}B-\alpha_2 A^{n-2}B-\ldots-\alpha_{n-1}AB-\alpha_n B. \tag{33}$$

Multiplying both parts of equality (33) by a non-zero string vector $\eta^T$, we have:

$$\eta^T A^n B=-\alpha_1\eta^T A^{n-1}B-\alpha_2\eta^T A^{n-2}B-\ldots-\alpha_{n-1}\eta^T AB-\alpha_n\eta^T B. \tag{34}$$

Since equalities (30) imply that all terms on the right-hand side of (34) are equal to zero, then $\eta^T A^n B=0$. Now multiply equality (33) on the left by $\eta^T A$ and get $\eta^T A^{n+1}B=0$. Continuing in the same way, we obtain equalities (31) for all-natural ones $j$.

Assume that the system (27) is globally controlled. Then the system of integral equations $\int_{t_0}^{t_1} e^{A\left(\tau-t_0\right)}Bu(\tau)\,d\tau=y$ must have continuous solutions for any fixed vector. In particular, there is a solution $u=\widetilde{u}(\tau)$ also for $y=\eta$, that is, the equality holds

$$\int_{t_0}^{t_1} e^{A(\tau-t_0)}B\widetilde{u}(\tau)\,d\tau=\eta \tag{35}$$

Given (29), let's write equality (35) in the form

$$\int_{t_0}^{t_1}\left[B+AB\left(t-t_0\right)+\frac{1}{2!}A^2B\left(t-t_0\right)^2+\frac{1}{3!}A^3B\left(t-t_0\right)^3+\ldots\right]\widetilde{u}(\tau)\,d\tau=\eta.$$

We multiply both parts of the obtained equality from the left by the row vector $\eta^T$. At the same time, the left part will turn into 0, because all terms of the expression under the sign of the integral will turn into 0, and the right will be $\|\eta\|^2\neq 0$. The resulting contradiction proves the necessity of condition (28). The theorem is proved.

***Note***. For the linear system (27) with constant coefficients, it does not matter on which segment the global controllability is considered. If the system (27) is globally controllable on the interval $[0,\,1]$, for example, then it will be globally controllable on any interval $\left[t_0,\,t_1\right]$.

**Example 1.** Prove that the system

$$\begin{cases}\dot{x}_1=x_2+x_3+b_1u,\\ \dot{x}_2=x_1+x_3+b_2u,\\ \dot{x}_3=x_1+x_2+b_3u\end{cases}$$

with a scalar control function $u=u(t)$ cannot be globally controlled, no matter what the constant coefficients are $b_i,\ i=1,\ 2,\ 3$.

***Proof.*** Let's write down the matrices

$$A=\begin{pmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{pmatrix},\ B=\begin{pmatrix}b_1\\ b_2\\ b_3\end{pmatrix}$$

and calculate the control matrix

$$W=\left(B, AB, A^2 B\right)=\begin{pmatrix}b_1 & b_2+b_3 & 2b_1+b_2+b_3\\ b_2 & b_1+b_3 & b_1+2b_2+b_3\\ b_3 & b_1+b_2 & b_1+b_2+2b_3\end{pmatrix}.$$

The determinant of this matrix is equal to zero for any volume $b_i,\ i=1,\ 2,\ 3$, therefore, $rangW<3$. That is, the necessary condition of global controllability is not fulfilled and therefore the system is not globally controllable.

Note that if the scalar control function is replaced by a vector control function in this system, the system will become globally controllable. For example, the system

$$\begin{cases}\dot{x}_1=x_2+x_3+u_1,\\ \dot{x}_2=x_1+x_3+u_2,\\ \dot{x}_3=x_1+x_2\end{cases}$$

is globally managed.

## Conclusions

Cyber security is one of the components of the state's information security. Therefore, an important task is the control of protection systems. The paper analyses the conditions of global controllability of stationary and non-stationary linear systems of differential equations used to model dynamic processes in the information space. In particular, the necessary and sufficient conditions for the global controllability of linear models in cybersecurity problems are presented. These conditions ensure effective control of dynamic processes even in the presence of a complex system structure.

The obtained results can be used to develop optimal strategies for managing information security in the context of the dynamic development of cyber threats. They also contribute to improving the efficiency of critical information systems protection. An important area for further research is the adaptation of the developed approaches to nonlinear systems, as well as the consideration of stochastic factors that can significantly affect the dynamics of processes in the information space.

## Declaration on Generative AI

While preparing this work, the authors used the AI programs Grammarly Pro to correct text grammar and Strike Plagiarism to search for possible plagiarism. After using this tool, the authors reviewed and edited the content as needed and took full responsibility for the publication's content.

## References


[1] S. Shevchenko, Y. Zhdanova, O. Kryvytska, H. Shevchenko, Fuzzy cognitive MAPPING as a scenario approach for information security risk analysis, in: Cybersecurity Providing in Information and Telecommunication Systems II, vol. 3826, 2024, 356–362.

[2] Á. Martín del Rey, G. Rodríguez Sánchez, Application to cybersecurity of the stability theory of the systems of ordinary differential equations, Geometry, Algebra and Applications: From Mechanics to Cryptography, 161 (2016). doi:10.1007/978-3-319-32085-4_14

[3] W. Liu, S. Zhong, Web malware spread modelling and optimal control strategies, Sci. Rep. 7 (2017). doi:10.1038/srep42308

[4] J. Diamantino Hernández Guillén, Á. Martín del Rey, R. Casado Vara, On the optimal control of a malware propagation model, Math. 8(9) (2020). doi:10.3390/math8091518

[5] I. V. Kononovich, Dynamics of the number of information security incidents, Inform. Math. Methods in Simulation, 4(1) (2014) 35–43.

[6] I. Kononovich, D. Mayevskiy, R. Podobniy, Models of system of the cibersecurity providing with delay of reaction on incidents, Inf. Math. Methods Simul. 5(4) (2015) 339–346.

[7] S. Yevseiev, et al., Modeling the Protection of Personal Data from Trust and the Amount of Information on Social Networks, EUREKA: Phys. Eng.1 (2021) 24–31. doi:10.21303/2461-4262.2021.001615

[8] S. Yevseiev, et. al. Development of a method for assessing the security of cyber-physical systems based on the Lotka–Volterra model, Eastern-European J. Enterp. Technol. 5(9) (2021) (113). doi:10.15587/1729-4061.2021.241638

[9] S. Shevchenko, et. al., Conflict analysis in the “subject-to-subject” security system, in: Cybersecurity Providing in Information and Telecommunication Systems, vol. 3421, 2023, 56–66.

[10] I. Karpovich, O. Hladka, A. Tymrakevych, Methodology for assessing the impact of information threats in the conditions of information confrontation, Electron Prof. Sci. J. Cybersecur. Educ. Sci. Tech. 2(26) (2024) 212–222. doi:10.28925/2663-4023.2024.24.8189

[11] J. B. Baillieul, J. C. Willems, Mathematical controltheory, NewYork, Springer-Verlag, 1999.

[12] F. Fairman, Linear control theory: The state space approach, John Wiley and Sons, 1998.

[13] D. U. Kirk, Optimal control theory. An Introduction, Mineola, New York:,Dover, 2004.

[14] V. F. Krotov, Global methods in optimal control theory. New York, Marcel Dekker, 1996.

[15] P. Chen, S. Cheng, K. Chen, Optimal control of epidemic information dissemination over networks, in: IEEE Trans. Cybern. 44(12), 2014, 2316–2328. doi:10.1109/TCYB.2014.2306781

[16] P. Sungu Ngoy, K. Musumbu, D. Kioi Gathungu, A mathematical modeling approach in cybersecurity using deep neural learning, Int. J. Adv. Res. Sci. Eng. Technol. 8(60) (2021) 17570–17587.

[17] V. Negodenko, Application of the mathematical catastrophe theory to ensure the stability of the information security management system, Electron Prof. Sci. J. Cybersecur. Educ. Sci. Tech. 2(26), (2024). 212–222. doi:10.28925/2663-4023.2024.26.692

[18] R. Syrotynskyi, et al., Methodology of network infrastructure analysis as part of migration to zero-trust architecture, in: Cyber Security and Data Protection, vol. 3800 (2024) 97–105.

[19] S. Gnatyuk, et al., Method for managing IT incidents in critical information infrastructure facilities, in: Cybersecurity Providing in Information and Telecommunication Systems II, vol. 3826 (2024) 326–333.

[20] R. E. Kalman, Y. C. Ho, K. S. Narendra, Controllability of linear dynamical systems, Contr. Diff. Eqs. 1 (1963) 189–213.

[21] M. F. Atiyah, I. G. MacDonald, Introduction to commutative algebra, Westview Press, 1969.